

Ground States of Quantum Electrodynamics with Cutoffs

Toshimitsu Takaesu

*Faculty of Science and Technology, Gunma University,
Gunma, 371-8510, Japan*

Abstract In this paper, we investigate a system of quantum electrodynamics with cut-offs. The total Hamiltonian is defined on a tensor product of a fermion Fock space and a boson Fock. It is shown that, under spatially localized conditions and momentum regularity conditions, the total Hamiltonian has a ground state for all values of coupling constants. In particular, its multiplicity is finite.

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1 Introduction

This article is concerned with a system of quantum electrodynamics with cutoffs. In quantum field theory, the interactions of charged particles and photons are described by quantum electrodynamics. We consider the system of a massive Dirac field coupled to a radiation field. The radiation field is quantized in the Coulomb gauge. In this system, the process of electron-positron pair production and annihilation occurs. We mathematically investigate the spectrum of the total Hamiltonian for the system. The Hilbert space for the system is defined by a tensor product of a fermion Fock space and boson Fock space, which is called a boson-fermion Fock space. The total Hamiltonian is given by

$$H_{\text{QED}} = H_{\text{D}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\text{rad}} + \kappa_{\text{I}} \sum_{j=1}^3 \int_{\mathbf{R}^3} \chi_{\text{I}}(\mathbf{x}) (\psi^\dagger(\mathbf{x}) \alpha^j \psi(\mathbf{x}) \otimes A_j(\mathbf{x})) d\mathbf{x} \\ + \kappa_{\text{II}} \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{\text{II}}(\mathbf{x}) \chi_{\text{II}}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} (\psi^\dagger(\mathbf{x}) \psi(\mathbf{x}) \psi^\dagger(\mathbf{y}) \psi(\mathbf{y}) \otimes \mathbb{1}) d\mathbf{x} d\mathbf{y}$$

on the Hilbert space. Here H_{D} and H_{rad} denote the energy Hamiltonians of the Dirac field and radiation field, respectively, $\psi(\mathbf{x})$ the Dirac field operator, $\mathbf{A}(\mathbf{x}) = (A_j(\mathbf{x}))_{j=1}^3$ the radiation field operator, $\alpha = (\alpha_j)_{j=1}^3$ 4×4 Dirac matrices, and $\chi_{\text{I}}(\mathbf{x})$ and $\chi_{\text{II}}(\mathbf{x})$ the spatial cutoffs. The constants $\kappa_{\text{I}} \in \mathbf{R}$ and $\kappa_{\text{II}} \in \mathbf{R}$ are called coupling constants. Ultraviolet cutoffs are imposed on $\psi(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$, respectively.

By making use of the spacial cutoffs and ultraviolet cutoffs, H_{QED} is self-adjoint operator on the Hilbert space, and the spectrum of H_{QED} is bounded from below. The main interest in this paper is the lower bound of the spectrum of H_{QED} . If the infimum of the spectrum of a self-adjoint operator is eigenvalue, the eigenvector is called ground state. The infimum of the spectrum of $H_0 = H_{\text{D}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\text{rad}}$ is eigenvalue, but it is embedded in continuous spectrum. This is because the radiation field is a massless field. It is not clear that H_{QED} has a ground state since the embedded

eigenvalue is not stable when interactions are turned on.

The ground state of H_{QED} for sufficiently small values of coupling constants was proven in [22]. The aim of this paper is to prove that H_{QED} has a ground state for all values of coupling constants. In particular, its multiplicity is finite. For the ground states of other QED models, Dimassi-Guillot [11] and Barbaroux-Dimassi-Guillot [7] investigated the system of the Dirac field in external potential coupled to the radiation field. They proved the existence of the ground state of the total Hamiltonian with generalized interactions for sufficiently small values of coupling constants. As far as we know, the existence of the ground states for the systems of a fermionic field coupled to a massless bosonic field, which include QED models, has not been proven for all values of coupling constants until now.

To prove the existence of the ground state of H_{QED} for all values of coupling constants, we apply the methods for systems of particles coupled bosonic fields. The spectral analysis and scattering theory for these systems, which include the non-relativistic QED models, have been progressed since the middle of '90s. The existence of the ground states was established by Arai-Hirokawa [3], Bach-Fröhlich-Sigal [5, 6], Gérard [14], Griesemer-Lieb-Loss [16], Lieb-Loss [20], Spohn [21] and many researchers. The strategy is as follows.

[1st Step] We introduce approximating Hamiltonians H_m , $m > 0$. Physically, $m > 0$ denotes the artificial mass of photon, and we call H_m a massive Hamiltonian. To prove the existence of ground states of H_m , we use partition of unity on Fock space, which was developed by Dereziński-Gérard [10]. We especially need the partitions of unity for both Dirac field and radiation field. By the partitions of unity and the Weyl sequence method, we prove that a positive spectral gap above the infimum of the spectrum exists for all values of coupling constants. From this, the existence of the ground states of H_m for all values of coupling constants follows.

[2nd Step] Let Ψ_m be the ground state of H_m , $m > 0$. Without loss of generality, we may assume that the Ψ_m is normalized. Then, there exists a subsequence of $\{\Psi_{m_j}\}_{j=1}^{\infty}$ with $m_{j+1} < m_j$, $j \in \mathbf{N}$, such that the weak limit of $\{\Psi_{m_j}\}_{j=1}^{\infty}$ exists. The key point is to show that the weak limit is non-zero vector. To prove this, we consider a combined method of Gerard [14] and Griesemer-Lieb-Loss in [16]. We use the electron positron derivative bounds and photon derivative bounds. To derive these bounds, the argument of the spatially localization is needed. For the spatially localized conditions, we suppose

$$\int_{\mathbf{R}^3} |\mathbf{x}| |\chi_I(\mathbf{x})| dx < \infty, \quad \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{|\chi_{II}(\mathbf{x}) \chi_{II}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|} |\mathbf{x}| d\mathbf{x} d\mathbf{y} < \infty.$$

In addition, We imposed momentum regularity conditions on the Dirac field and radiation field, which include the infrared regularity condition

$$\int_{\mathbf{R}^3} \frac{|\chi_{\text{rad}}(\mathbf{k})|^2}{|\mathbf{k}|^5} d\mathbf{k} < \infty.$$

We briefly review the results for the systems of fermionic fields coupled bosonic fields. For QED models, the Gell-Mann - Low formula of H_{QED} was obtained by Futakuchi-Usui [12]. For the Yukawa model, which is the system for a massive Dirac field interacting with a massive Klein-Gordon field, the existence of the ground state was proven in [23]. The spectral analysis for the weak interaction models has been analyzed, and refer to Barbaroux-Faupin-Guillot [8], Guillot [17] and the reference therein.

This paper is organized as follows. In section 2, full Fock spaces, fermion Fock spaces and boson Fock spaces are introduced, and Dirac field operators and radiation field operators are defined on a Fermion Fock space and boson Fock space, respectively. The total Hamiltonian is defined on a boson-fermion Fock space and the main theorem is stated. In Section 3, partitions of unity for the Dirac field and radiation field are investigated. Then the existence of the ground state of H_m is proven. In section 4, the derivative bounds for electrons-positrons and photons are derived. In Section 5, we give the proof of the main theorem.

2 Notations and Main Results

2.1 Fock Spaces

(i) Full Fock Space

The full Fock space over a complex Hilbert space \mathcal{Z} is defined by $\mathcal{F}(\mathcal{Z}) = \bigoplus_{n=0}^{\infty} (\otimes^n \mathcal{Z})$ where $\otimes^n \mathcal{Z}$ is the n fold tensor product of \mathcal{Z} . The Fock vacuum is defined by $\Omega = \{1, 0, 0, \dots\} \in \mathcal{F}(\mathcal{Z})$. Let $\mathcal{L}(\mathcal{Z})$ be the set which consists of all linear operators on \mathcal{Z} . The functor of $Q \in \mathcal{L}(\mathcal{Z})$ is defined by $\Gamma(Q) = \bigoplus_{n=0}^{\infty} (\otimes^n Q)$ and the second quantization of $T \in \mathcal{L}(\mathcal{Z})$ is given by $d\Gamma(T) = \bigoplus_{n=0}^{\infty} \tilde{T}^{(n)}$ with $\tilde{T}^{(n)} = \sum_{j=1}^n ((\otimes^{j-1} \mathbb{1}) \otimes T \otimes (\otimes^{n-j} \mathbb{1}))$. The number operator is defined by $N = d\Gamma(\mathbb{1})$.

(ii) Fermion Fock Space

The fermion Fock space over a complex Hilbert space \mathcal{X} is defined by $\mathcal{F}_f(\mathcal{X}) = \bigoplus_{n=0}^{\infty} (\otimes_a^n \mathcal{X})$ where $\otimes_a^n \mathcal{X}$ denotes the n -fold anti-symmetric tensor product of \mathcal{X} . The Fock vacuum is defined by $\Omega_f = \{1, 0, 0, \dots\} \in \mathcal{F}_f(\mathcal{X})$. Let T_f and Q_f be linear operators on \mathcal{X} . We set $d\Gamma_f(T_f) = d\Gamma(T_f)|_{\mathcal{F}_f(\mathcal{X})}$ and $\Gamma_f(Q_f) = \Gamma(Q_f)|_{\mathcal{F}_f(\mathcal{X})}$ where $X|_{\mathcal{M}}$ is the restriction of the operator X to the subspace \mathcal{M} . The number operator is defined by $N_f = d\Gamma_f(\mathbb{1})$. The creation operator $C^\dagger(f)$, $f \in \mathcal{X}$, is defined by $(C^\dagger(f)\Psi)^{(n)} = \sqrt{n} U_a^n(f \otimes \Psi^{(n-1)})$, $n \geq 1$, and $(C^\dagger(f)\Psi)^{(0)} = 0$ where U_a^n is the projection from $\otimes^n \mathcal{X}$ to $\otimes_a^n \mathcal{X}$. The annihilation operator $C(f)$ is defined by $C(f) = (C^\dagger(f))^*$ where X^* denotes the adjoint of the operator X . For each subspace $\mathcal{M} \subset \mathcal{X}$, the finite particle space $\mathcal{F}_f^{\text{fin}}(\mathcal{M})$ is defined by the linear hull of Ω_f and $C^\dagger(f_1), \dots, C^\dagger(f_n)\Omega_f$, $j = 1, \dots, n$, $n \in \mathbf{N}$. The creation and annihilation operators satisfy the canonical anti-commutation relations on $\mathcal{F}_f(\mathcal{X})$:

$$\{C(f), C^\dagger(f')\} = (f, f'), \quad \{C^\dagger(f), C^\dagger(f')\} = \{C(f), C(f')\} = 0, \quad f, f' \in \mathcal{X},$$

where $\{X, Y\} = XY + YX$.

(ii) Boson Fock Space

The boson Fock space over a complex Hilbert space \mathcal{Y} is defined by $\mathcal{F}_b(\mathcal{Y}) = \bigoplus_{n=0}^{\infty} (\otimes_s^n \mathcal{Y})$ where $\otimes_s^n \mathcal{Y}$ denotes the n -fold symmetric tensor product of \mathcal{Y} . The Fock vacuum is given by $\Omega_b = \{1, 0, 0, \dots\} \in \mathcal{F}(\mathcal{Y})$. Let T_b and Q_b be linear operators on \mathcal{Y} . Then we define $d\Gamma_b(T_b) = d\Gamma(T_b)|_{\mathcal{F}_b(\mathcal{Y})}$ and $\Gamma_b(Q) = \Gamma(Q_b)|_{\mathcal{F}_b(\mathcal{Y})}$. The number operator is defined by $N_f = d\Gamma_f(\mathbb{1})$. The creation operator $A^\dagger(g)$, $g \in \mathcal{Y}$, is defined by $(A^\dagger(g)\Phi)^{(n)} = \sqrt{n}U_s^n(f \otimes \Phi^{(n-1)})$, $n \geq 1$, and $(A^\dagger(g)\Phi)^{(0)} = 0$ where U_s^n is the projection from $\otimes^n \mathcal{Y}$ to $\otimes_s^n \mathcal{Y}$. The annihilation operator $A(f)$ is defined by $A(g) = (A^\dagger(g))^*$. The finite particle space $\mathcal{F}_b^{\text{fin}}(\mathcal{N})$ on the subspace $\mathcal{N} \subset \mathcal{Y}$ defined by the linear hull of Ω_f and $A^\dagger(g_1), \dots, A^\dagger(g_n)\Omega_f$, $j = 1, \dots, n$, $n \in \mathbf{N}$. The creation and annihilation operators satisfy the canonical commutation relations on $\mathcal{F}_b^{\text{fin}}(\mathcal{N})$:

$$[A(g), A^\dagger(g')] = (g, g'), \quad [A^\dagger(g), A^\dagger(g')] = [A(g), A(g')] = 0, \quad g, g' \in \mathcal{Y},$$

where $[X, Y] = XY - YX$.

2.2 Dirac field

Let $\mathcal{F}_{\text{Dir}} = \mathcal{F}_f(L^2(\mathbf{R}^3; \mathbf{C}^4))$. The energy Hamiltonian of the Dirac field is defined by

$$H_D = d\Gamma_f(\omega_M)$$

where $\omega_M(\mathbf{p}) = \sqrt{|\mathbf{p}|^2 + M^2}$, $\mathbf{p} \in \mathbf{R}^3$, and $M > 0$. Let $C^\dagger(t(f_1, \dots, f_4))$, $f_l \in L^2(\mathbf{R}^3)$, $l = 1, \dots, 4$, be the creation operator on $\mathcal{F}_{\text{Dirac}}$. For each $f \in L^2(\mathbf{R}^3)$, we set

$$\begin{aligned} b_{1/2}^\dagger(f) &= C^\dagger(t(f, 0, 0, 0)), & b_{-1/2}^\dagger(f) &= C^\dagger(t(0, f, 0, 0)), \\ d_{1/2}^\dagger(f) &= C^\dagger(t(0, 0, f, 0)), & d_{-1/2}^\dagger(f) &= C^\dagger(t(0, 0, 0, f)). \end{aligned}$$

We define $b_s(f)$ and $d_s(g)$ by the conjugate of $b_s^\dagger(f)$ and $d_s^\dagger(g)$, respectively. Then the canonical anti-commutation relations

$$\{b_s(f), b_{s'}^\dagger(g)\} = \{d_s(f), d_{s'}^\dagger(g)\} = \delta_{s,s'}(f, g), \quad (1)$$

$$\{b_s(f), b_{s'}(g)\} = \{d_s(f), d_{s'}(g)\} = \{b_s(g), d_{s'}^\dagger(g)\} = 0, \quad (2)$$

are satisfied and it holds that

$$\|b_s(f)\| = \|b_s^\dagger(f)\| = \|f\|, \quad \|d_s(g)\| = \|d_s^\dagger(g)\| = \|g\|. \quad (3)$$

Let $h_D(\mathbf{p}) = \alpha \cdot \mathbf{p} + M\beta$ be the Fourier transformed Dirac operator with 4×4 Dirac matrices $\alpha = (\alpha^j)_{j=1}^3$ and β . Let $\mathbf{S}(\mathbf{p}) = \mathbf{S} \cdot \mathbf{p}$, $\mathbf{p} \in \mathbf{R}^3$, where $\mathbf{S} = -\frac{i}{4}\alpha \wedge \alpha$ is the spin angular momentum. The spinors $u_s(\mathbf{p}) = (u_s^l(\mathbf{p}))_{l=1}^4$ and $v_s(\mathbf{p}) = (v_s^l(\mathbf{p}))_{l=1}^4$ are function which satisfy the following :

$$(D.1) \quad h_D(\mathbf{p})u_s(\mathbf{p}) = E_M(\mathbf{p})u_s(\mathbf{p}), \quad h_D(\mathbf{p})v_s(\mathbf{p}) = -E_M(\mathbf{p})v_s(\mathbf{p}),$$

$$(D.2) \quad S(\mathbf{p})u_s(\mathbf{p}) = s|\mathbf{p}|u_s(\mathbf{p}), \quad S(\mathbf{p})v_s(\mathbf{p}) = s|\mathbf{p}|v_s(\mathbf{p}),$$

$$(D.3) \quad \sum_{l=1}^4 u_s^l(\mathbf{p})^* u_{s'}^l(\mathbf{p}') = \sum_{l=1}^4 v_s^l(\mathbf{p})^* v_{s'}^l(\mathbf{p}') = \delta_{s,s'}, \quad \sum_{l=1}^4 u_s^l(\mathbf{p})^* v_{s'}^l(\mathbf{p}') = 0.$$

Remark 2.1 We review the example of spinors in the standard representation (see [24] ; Section 1). The Pauli matrices are defined by $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then, the Dirac matrices are $\alpha = \begin{pmatrix} O & \sigma \\ \sigma & O \end{pmatrix}$, $\beta = \begin{pmatrix} \mathbb{1} & O \\ O & -\mathbb{1} \end{pmatrix}$, and the spin angular momentum is $\mathbf{S} = \frac{1}{2} \begin{pmatrix} \sigma & O \\ O & \sigma \end{pmatrix}$. Let $O_{\text{SR}} = \{\mathbf{p} = (p^1, p^2, p^3) \in \mathbf{R}^3 \mid |\mathbf{p}| - p^3 = 0\}$. We see that the Lebesgue measure of O_{SR} is zero. We set

$$\eta_+(\mathbf{p}) = \begin{cases} \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| - p^3)}} \begin{pmatrix} p^1 - ip^2 \\ |\mathbf{p}| - p^3 \end{pmatrix}, & \mathbf{p} \notin O_{\text{SR}}, \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \mathbf{p} \in O_{\text{SR}}, \end{cases} \quad \eta_-(\mathbf{p}) = \begin{cases} \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| - p^3)}} \begin{pmatrix} p^3 - |\mathbf{p}| \\ p^1 + ip^2 \end{pmatrix}, & \mathbf{p} \notin O_{\text{SR}}, \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \mathbf{p} \in O_{\text{SR}}. \end{cases}$$

Let

$$u_{\pm 1/2}(\mathbf{p}) = \begin{pmatrix} \lambda_+(\mathbf{p})\eta_{\pm}(\mathbf{p}) \\ \pm \lambda_-(\mathbf{p})\eta_{\pm}(\mathbf{p}) \end{pmatrix}, \quad v_{\pm 1/2}(\mathbf{p}) = \begin{pmatrix} \mp \lambda_-(\mathbf{p})\phi_{\pm}(\mathbf{p}) \\ \pm \lambda_+(\mathbf{p})\eta_{\pm}(\mathbf{p}) \end{pmatrix},$$

with $\lambda_{\pm}(\mathbf{p}) = \frac{1}{\sqrt{2}} \sqrt{1 \pm ME_M(\mathbf{p})^{-1}}$. Here note that u_s and v_s satisfy $u_s, v_s \in \oplus^4(C^1(\mathbf{R}^3 \setminus O_{\text{SR}}))$.

The Dirac field operator $\psi(\mathbf{x}) = {}^t(\psi_1(\mathbf{x}), \dots, \psi_4(\mathbf{x}))$ is defined by

$$\psi_l(\mathbf{x}) = \sum_{s=\pm 1/2} \left(b_s(f_{s,\mathbf{x}}^l) + d_s^\dagger(g_{s,\mathbf{x}}^l) \right), \quad l = 1, \dots, 4,$$

where $f_{s,\mathbf{x}}^l(\mathbf{p}) = f_s^l(\mathbf{p})e^{-\mathbf{p} \cdot \mathbf{x}}$ with $f_s^l(\mathbf{p}) = \frac{1}{\sqrt{(2\pi)^3}} \chi_D(\mathbf{p}) u_s^l(\mathbf{p})$ and $g_{s,\mathbf{x}}^l(\mathbf{p}) = g_s^l(\mathbf{p})e^{-\mathbf{p} \cdot \mathbf{x}}$ with $g_s^l(\mathbf{p}) = \frac{1}{\sqrt{(2\pi)^3}} \chi_D(\mathbf{p}) \tilde{v}_s^l(\mathbf{p})$ and $\tilde{v}_s^l(\mathbf{p}) = v_s^l(-\mathbf{p})$. Here χ_D satisfy the following condition.

(A.1 ; Ultraviolet Cutoff for Dirac Field)

$$\int_{\mathbf{R}^3} |\chi_D(\mathbf{p})|^2 d\mathbf{p} < \infty.$$

Then it holds that

$$\|\psi_l(\mathbf{x})\| \leq c_D^l, \quad (4)$$

where $c_D^l = \frac{1}{\sqrt{(2\pi)^3}} \sum_{s=\pm 1/2} (\|\chi_D u_s^l\| + \|\chi_D \tilde{v}_s^l\|)$, $l = 1, \dots, 4$.

2.3 Radiation Field in the Coulomb Gauge

Let $\mathcal{F}_{\text{rad}} = \mathcal{F}_{\text{b}}(\oplus_{r=1,2} L^2(\mathbf{R}^3))$. The free Hamiltonian is defined by

$$H_{\text{rad}} = d\Gamma_{\text{b}}(\omega)$$

where $\omega(\mathbf{k}) = |\mathbf{k}|$, $\mathbf{k} \in \mathbf{R}^3$. Let $A^*(h_1, h_2)$, $h_r \in L^2(\mathbf{R}^3)$, $r = 1, 2$, be the creation operators on \mathcal{F}_{rad} . Let

$$a_1^\dagger(h) = A((h, 0)), \quad a_2^\dagger(h) = A((0, h)), \quad h \in L^2(\mathbf{R}^3),$$

and $a_r(h') = (a^\dagger(h'))^*$, $h' \in L^2(\mathbf{R}^3)$, $r = 1, 2$. The creation operators and annihilation operators satisfy the canonical commutation relations

$$[a_r(h), a_{r'}^\dagger(h')] = \delta_{r,r'}(h, h'), \quad (5)$$

$$[a_r(h), a_{r'}(h')] = [a_r^\dagger(h'), a_{r'}^\dagger(h')] = 0, \quad (6)$$

on $\mathcal{F}_{\text{rad}}^{\text{fin}}(\mathcal{M})$ where \mathcal{M} is a subspace of $\oplus_{r=1,2} L^2(\mathbf{R}^3)$. For all $h \in \mathcal{D}(\omega^{-1/2})$, it follows that

$$\|a_r(h)(H_{\text{rad}} + 1)^{-1/2}\| \leq \left\| \frac{h}{\sqrt{\omega}} \right\|, \quad \|a_r^\dagger(h)(H_{\text{rad}} + 1)^{-1/2}\| \leq \left\| \frac{h}{\sqrt{\omega}} \right\| + \|h\|. \quad (7)$$

The polarization vectors $\mathbf{e}_r(\mathbf{k}) = (e_r^j(\mathbf{k}))$, $r = 1, 2$, satisfy the following relations.

$$(R.1) \quad \mathbf{e}_r(\mathbf{k}) \cdot \mathbf{e}_{r'}(\mathbf{k}) = 0, \quad \mathbf{e}_r(\mathbf{k}) \cdot \mathbf{k} = 0, \quad \mathbf{k} \in \mathbf{R}^3 \setminus \{\mathbf{0}\}.$$

Remark 2.2 We check the example of the polarization vectors. For all $\mathbf{k} \in \mathbf{R}^3 \setminus \{\mathbf{0}\}$, we set

$$\mathbf{e}_1(\mathbf{k}) = \frac{1}{\sqrt{(k^1)^2 + (k^2)^2}} \begin{pmatrix} -k^2 \\ k^1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2(\mathbf{k}) = \frac{1}{|\mathbf{k}| \sqrt{k_1^2 + k_2^2}} \begin{pmatrix} k^1 k^3 \\ k^2 k^3 \\ -(k^1)^2 - (k^2)^2 \end{pmatrix}.$$

Then (R.1) is satisfied. Here it is noted that $\mathbf{e}_r \in \oplus^3 (C^1(\mathbf{R}^3 \setminus \{\mathbf{0}\}))$, $r = 1, 2$.

The radiation field operator $\mathbf{A}(\mathbf{x}) = (A_j(\mathbf{x}))_{j=1}^3$ is defined by

$$A_j(\mathbf{x}) = \sum_{r=1,2} \left(a_r(h_{r,\mathbf{x}}^j) + a_r^\dagger(h_{r,\mathbf{x}}^j) \right)$$

where $h_{r,\mathbf{x}}^j(\mathbf{k}) = h_r^j(\mathbf{k})e^{-\mathbf{k} \cdot \mathbf{x}}$ with $h_r^j(\mathbf{k}) = \frac{1}{\sqrt{(2\pi)^3}} \frac{\chi_{\text{rad}}(\mathbf{k}) e_r^j(\mathbf{k})}{\sqrt{2\omega(\mathbf{k})}}$, and χ_{rad} satisfy the following condition.

(A.2 : Ultraviolet Cutoff for Radiation Field)

$$\int_{\mathbf{R}^3} \frac{|\chi_{\text{rad}}(\mathbf{k})|^2}{\omega(\mathbf{k})^l} d\mathbf{k} < \infty, \quad l = 1, 2.$$

Then

$$\|A_j(\mathbf{x})(H_{\text{rad}} + 1)^{-1/2}\| \leq c_{\text{rad}}^j \quad (8)$$

where $c_{\text{rad}}^j = \frac{1}{\sqrt{(2\pi)^3}} \sum_{r=1,2} \left(\sqrt{2} \left\| \frac{\chi_{\text{rad}} e_r^j}{\omega} \right\| + \left\| \frac{\chi_{\text{rad}} e_r^j}{\sqrt{2\omega}} \right\| \right).$

2.4 Total Hamiltonian and Main Theorem

We define the system of the Dirac field interacting with the radiation field. The Hilbert space for the system is defined by $\mathcal{F}_{\text{QED}} = \mathcal{F}_{\text{Dirac}} \otimes \mathcal{F}_{\text{rad}}$. The free Hamiltonian is defined by

$$H_0 = H_D \otimes \mathbb{1}_{\text{rad}} + \mathbb{1}_D \otimes H_{\text{rad}}$$

on the domain $\mathcal{D}(H_0) = \mathcal{D}(H_D \otimes \mathbb{1}_{\text{rad}}) \cap \mathcal{D}(\mathbb{1}_D \otimes H_{\text{rad}})$. To define the interactions, we introduce spatial cutoff χ_I and χ_{II} , which satisfy the condition below.

(A.3 : Spatial Cutoff)

$$\int_{\mathbf{R}^3} |\chi_I(\mathbf{x})| d\mathbf{x} < \infty, \quad \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{|\chi_{II}(\mathbf{x})\chi_{II}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|} d\mathbf{x} d\mathbf{y} < \infty.$$

First we define a functional on $\mathcal{F}_{\text{QED}} \times \mathcal{F}_{\text{QED}}$ by

$$\ell_I(\Phi, \Psi) = \sum_{j=1}^3 \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) (\Phi, (\psi^\dagger(\mathbf{x}) \alpha^j \psi(\mathbf{x}) \otimes A_j(\mathbf{x})) \Psi), \quad \Phi \in \mathcal{F}_{\text{QED}}, \Psi \in \mathcal{D}(H_0),$$

where $\psi^\dagger(\mathbf{x}) = (\psi_1(\mathbf{x})^*, \dots, \psi_4(\mathbf{x})^*)$. We see that

$$|\ell_I(\Phi, \Psi)| \leq \left(\int_{\mathbf{R}^3} |\chi_I(\mathbf{x})| d\mathbf{x} \right) \sum_{j=1}^3 \sum_{l, l'=1}^4 |\alpha_{l, l'}^j| c_D^l c_D^{l'} c_{\text{rad}}^j \|\Phi\| \|\mathbb{1}_D \otimes (H_{\text{rad}} + 1)^{1/2} \Psi\|.$$

By the Riesz representation theorem, we can define the operator H_I which satisfy $(\Phi, H_I \Psi) = \ell_I(\Phi, \Psi)$ and

$$\|H_I \Psi\| \leq c_I \|\mathbb{1}_D \otimes (H_{\text{rad}} + 1)^{1/2} \Psi\|, \quad (9)$$

where $c_I = \|\chi_I\|_{L^1} \sum_{j=1}^3 \sum_{l, l'=1}^4 |\alpha_{l, l'}^j| c_D^l c_D^{l'} c_{\text{rad}}^j$. By the spectral decomposition theorem, it is proven that for all $\varepsilon > 0$,

$$\|H_I \Psi\| \leq c_I \varepsilon \|H_0 \Psi\| + c_I \left(\frac{1}{2\varepsilon} + 1 \right) \|\Psi\|. \quad (10)$$

Next we define a functional on $\mathcal{F}_{\text{QED}} \otimes \mathcal{F}_{\text{QED}}$ by

$$\ell_{II}(\Phi, \Psi) = \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{II}(\mathbf{x})\chi_{II}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} (\Phi, (\psi^\dagger(\mathbf{x}) \psi(\mathbf{x}) \psi^\dagger(\mathbf{y}) \psi(\mathbf{y}) \otimes \mathbb{1}_{\text{rad}}) \Psi) d\mathbf{x} d\mathbf{y}, \quad \Phi, \Psi \in \mathcal{F}_{\text{QED}}.$$

We see that

$$|\ell_{II}(\Phi, \Psi)| \leq \left(\int_{\mathbf{R}^3 \times \mathbf{R}^3} \left| \frac{\chi_{II}(\mathbf{x})\chi_{II}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \right| d\mathbf{x} d\mathbf{y} \right) \sum_{l, l'=1}^4 (c_D^l c_D^{l'})^2 \|\Phi\| \|\Psi\|.$$

Then, by the Riesz representation theorem, we can define an operator H_{II} satisfying $(\Phi, H_{II} \Psi) = \ell_{II}(\Phi, \Psi)$ and

$$\|H_{II}\| \leq c_{II}, \quad (11)$$

where $c_{\text{II}} = \left\| \frac{\chi_{\text{II}}(\mathbf{x})\chi_{\text{II}}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \right\|_{L^1} \sum_{l,l'=1}^4 (c_{\text{D}}^l c_{\text{D}}^{l'})^2$. By (10) and (11), it holds that

$$\|(\kappa_{\text{I}} H_{\text{I}} + \kappa_{\text{II}} H_{\text{II}})\Psi\| \leq c_{\text{I}} \kappa_{\text{I}} \varepsilon \|H_0 \Psi\| + \left(c_{\text{I}} \kappa_{\text{I}} \left(\frac{1}{2\varepsilon} + 1 \right) + c_{\text{II}} \kappa_{\text{II}} \right) \|\Psi\|.$$

Then the Kato-Rellich theorem yields that that H_{QED} is self-adjoint on $\mathcal{D}(H_0)$ and essentially self-adjoint on any core of H_0 . Hence, in particular, H_{QED} is essentially self-adjoint on

$$\mathcal{D}_0 = \mathcal{F}_{\text{Dir}}^{\text{fin}} \mathcal{D}(\omega_M) \hat{\otimes} \mathcal{F}_{\text{rad}}^{\text{fin}}(\mathcal{D}(\omega))$$

where $\hat{\otimes}$ denotes the algebraic tensor product.

To prove the existence of the ground state of H_{QED} , we suppose additional conditions below.

(A.4 : Spatial Localization)

$$\int_{\mathbf{R}^3} |\mathbf{x}| |\chi_{\text{I}}(\mathbf{x})| d\mathbf{x} < \infty, \quad \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{|\chi_{\text{II}}(\mathbf{x})\chi_{\text{II}}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|} |\mathbf{x}| d\mathbf{x} d\mathbf{y} < \infty.$$

(A.5 : Momentum Regularity Condition for Dirac Field)

There exists a subset $O_{\text{D}} \subset \mathbf{R}^3$ with Lebesgue measure zero such that $u_s, v_s \in \oplus^4 (C^1(\mathbf{R}^3 \setminus O_{\text{D}}))$, $s = \pm 1/2$. $\chi_{\text{D}} \in C^1(\mathbf{R}^3)$, and it satisfies that

$$\int_{\mathbf{R}^3} |\partial_{p^v} \chi_{\text{D}}(\mathbf{p})|^2 d\mathbf{p} < \infty, \quad \int_{\mathbf{R}^3} |\chi_{\text{D}}(\mathbf{p}) \partial_{p^v} u_s^l(\mathbf{p})|^2 d\mathbf{p} < \infty, \quad \int_{\mathbf{R}^3} |\chi_{\text{D}}(\mathbf{p}) \partial_{p^v} v_s^l(-\mathbf{p})|^2 d\mathbf{p} < \infty,$$

for all $v = 1, 2, 3$, $l = 1, \dots, 4$, $s = \pm 1/2$.

(A.6 : Momentum Regularity Condition for Radiation Field)

There exists a subset $O_{\text{rad}} \subset \mathbf{R}^3$ with Lebesgue measure zero such that $\mathbf{e}_r \in \oplus^3 (C^1(\mathbf{R}^3 \setminus O_{\text{rad}}))$, $r = 1, 2$, where $O_{\text{rad}} \cdot \chi_{\text{rad}} \in C^1(\mathbf{R}^3)$ and it satisfies that

$$\int_{\mathbf{R}^3} \frac{|\chi_{\text{rad}}(\mathbf{k})|^2}{|\mathbf{k}|^5} d\mathbf{k} < \infty, \quad \int_{\mathbf{R}^3} \frac{|\partial_{k^v} \chi_{\text{rad}}(\mathbf{k})|^2}{|\mathbf{k}|^3} d\mathbf{k} < \infty, \quad \int_{\mathbf{R}^3} \frac{|\chi_{\text{rad}}(\mathbf{k}) \partial_{k^v} e_r^j(\mathbf{k})|^2}{|\mathbf{k}|^3} d\mathbf{k} < \infty,$$

for all $v = 1, 2, 3$, $j = 1, 2, 3$, $r = 1, 2$.

Remark 2.3 Examples of O_{D} and O_{rad} in (A.5) and (A.6) are as follows. In the case of the standard representation, $O_{\text{D}} = O_{\text{SR}}$ where O_{SR} is defined in Remark 2.1. For the polarization vectors considered in Remark 2.2, $O_{\text{rad}} = \{\mathbf{0}\}$.

The main theorem in this paper is as follows.

Theorem 2.1 (Existence of a Ground State)

Suppose (A.1) - (A.6). Then H_{QED} has a ground state for all values of coupling constants. In particular, its multiplicity is finite.

3 Ground States of Massive case

In this section, we consider a massive Hamiltonian defined by

$$H_m = H_D \otimes \mathbb{1} + \mathbb{1} \otimes H_{\text{rad},m} + \kappa_I H_I + \kappa_{II} H_{II},$$

where $H_{\text{rad},m} = d\Gamma_b(\omega_m)$ with $\omega_m(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$, $m > 0$.

3.1 Fock Spaces on Direct Sum of Hilbert Spaces

We review basic properties of Fock spaces on direct sum of Hilbert spaces. These are useful for constructing partitions of unity on Fock spaces (see, Dereziński-Gérard [10]).

(i) Full Fock Space on $\mathcal{Z} \oplus \mathcal{Z}$

Let $Z = \begin{bmatrix} Z_0 \\ Z_\infty \end{bmatrix}$, $Z_0, Z_\infty \in \mathcal{L}(\mathcal{Z})$, where \mathcal{Z} is a complex Hilbert space. We consider $Z = \begin{bmatrix} Z_0 \\ Z_\infty \end{bmatrix}$ is an operator $\mathcal{Z} \rightarrow \mathcal{Z} \oplus \mathcal{Z}$ which acts for

$$Zh = \begin{bmatrix} Z_0 h \\ Z_\infty h \end{bmatrix}, \quad h \in \mathcal{D}(Z_0) \cap \mathcal{D}(Z_\infty).$$

Let $J = \begin{bmatrix} J_0 \\ J_\infty \end{bmatrix}$, $J_0, J_\infty \in \mathcal{L}(\mathcal{Z})$ and $B = \begin{bmatrix} B_0 \\ B_\infty \end{bmatrix}$, $B_0, B_\infty \in \mathcal{L}(\mathcal{Z})$. We define $d\Gamma(J, B) : \mathcal{F}(\mathcal{Z}) \rightarrow \mathcal{F}(\mathcal{Z}) \oplus \mathcal{F}(\mathcal{Z})$ by

$$d\Gamma(J, B) = \oplus_{n=0}^{\infty} \left(\sum_{j=1}^n (\otimes^{j-1} J) \otimes B \otimes (\otimes^{n-j} J) \right).$$

If B_0 and B_∞ are bounded, and $J_0^* J_0 + J_\infty^* J_\infty \leq 1$, it holds that

$$\|d\Gamma(J, B)(N+1)^{-1}\| \leq \sqrt{\|B_0\|^2 + \|B_\infty\|^2}. \quad (12)$$

Let $T \in \mathcal{L}(\mathcal{Z})$. Then it holds that

$$\Gamma(J)d\Gamma(T) = d\Gamma\left(\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}\right)\Gamma(J) + d\Gamma(J, \tilde{\text{ad}}_T(J)), \quad (13)$$

where $\tilde{\text{ad}}_T(J) : \mathcal{Z} \rightarrow \mathcal{Z} \oplus \mathcal{Z}$ is defined by

$$\tilde{\text{ad}}_T(J)h = \begin{bmatrix} [T, J_0]h \\ [T, J_\infty]h \end{bmatrix}, \quad h \in \mathcal{D}([T, J_0]) \cap \mathcal{D}([T, J_\infty]).$$

(ii) Fermion Fock Space on $\mathcal{X} \oplus \mathcal{X}$

Let \mathcal{X} be a complex Hilbert space. Let $J_f = \begin{bmatrix} J_f^0 \\ J_f^\infty \end{bmatrix}$, $J_f^0, J_f^\infty \in \mathcal{L}(\mathcal{X})$ and $B_f = \begin{bmatrix} B_f^0 \\ B_f^\infty \end{bmatrix}$, $B_f^0, B_f^\infty \in$

$\mathcal{L}(\mathcal{X})$. We set $d\Gamma_f(J_f, B_f) = d\Gamma(J_f, B_f)|_{\mathcal{F}_f(\mathcal{X})}$. Suppose that B_f^0 and B_f^∞ are bounded, and $(J_f^0)^*J_f^0 + (J_f^\infty)^*J_f^\infty \leq 1$. By (12), it holds that

$$\|d\Gamma_f(J_f, B_f)(N_f + 1)^{-1}\| \leq \sqrt{\|B_f^0\|^2 + \|B_f^\infty\|^2}. \quad (14)$$

Let $T_f \in \mathcal{L}(\mathcal{X})$. From (13), it holds that

$$\Gamma_f(J_f)d\Gamma_f(T_f) = d\Gamma_f\left(\begin{bmatrix} T_f & 0 \\ 0 & T_f \end{bmatrix}\right)\Gamma_f(J_f) + d\Gamma_f(J_f, \tilde{\text{ad}}_{T_f}(J_f)). \quad (15)$$

Let $C(f)$ and $C^\dagger(f)$, $f \in \mathcal{X}$, be the annihilation and creation operators on $\mathcal{F}_f(\mathcal{X})$, respectively. Then it follows that

$$\Gamma_f(J_f)C(f) = C\left(\begin{bmatrix} f \\ 0 \end{bmatrix}\right)\Gamma_f(J_f) + \Gamma_f(J_f)C\left((1 - (J_f^0)^*)f\right), \quad (16)$$

$$\Gamma_f(J_f)C^\dagger(f) = C^\dagger\left(\begin{bmatrix} f \\ 0 \end{bmatrix}\right)\Gamma_f(J_f) + C^\dagger\left(\begin{bmatrix} J_f^0 - 1 \\ J_f^\infty \end{bmatrix}f\right)\Gamma_f(J_f). \quad (17)$$

(iii) Boson Fock Space on $\mathcal{Y} \oplus \mathcal{Y}$

Let \mathcal{Y} be a complex Hilbert space. Let $J_b = \begin{bmatrix} J_b^0 \\ J_b^\infty \end{bmatrix}$, $J_b^0, J_b^\infty \in \mathcal{L}(\mathcal{Y})$ and $B_b = \begin{bmatrix} B_b^0 \\ B_b^\infty \end{bmatrix}$, $B_b^0, B_b^\infty \in \mathcal{L}(\mathcal{Y})$. We define $d\Gamma_b(J_b, B_b) = d\Gamma(J_b, B_b)|_{\mathcal{F}_b(\mathcal{Y})}$. Assume that B_b^0 and B_b^∞ are bounded, and $(J_b^0)^*J_b^0 + (J_b^\infty)^*J_b^\infty \leq 1$. By (12), it follows that

$$\|d\Gamma_b(J_b, B_b)(N_b + 1)^{-1}\| \leq \sqrt{\|B_b^0\|^2 + \|B_b^\infty\|^2}. \quad (18)$$

Let $T_b \in \mathcal{L}(\mathcal{Y})$. Then (13) yields that

$$\Gamma_b(J_b)d\Gamma_b(T_b) = d\Gamma_b\left(\begin{bmatrix} T_b & 0 \\ 0 & T_b \end{bmatrix}\right)\Gamma_b(J_b) + d\Gamma_b(J_b, \tilde{\text{ad}}_{T_b}(J_b)). \quad (19)$$

Let $A(g)$ and $A^\dagger(g)$, $g \in \mathcal{Y}$, be the annihilation and creation operators on $\mathcal{F}_b(\mathcal{Y})$, respectively. Then it follows that

$$\Gamma_b(J_b)A(g) = A\left(\begin{bmatrix} g \\ 0 \end{bmatrix}\right)\Gamma_b(J_b) + \Gamma_b(J_b)A\left((1 - (J_b^0)^*)g\right), \quad (20)$$

$$\Gamma_b(J_b)A^\dagger(g) = A^\dagger\left(\begin{bmatrix} g \\ 0 \end{bmatrix}\right)\Gamma_b(J_b) + A^\dagger\left(\begin{bmatrix} J_b^0 - 1 \\ J_b^\infty \end{bmatrix}g\right)\Gamma_b(J_b). \quad (21)$$

3.2 Partition of Unity for the Dirac Field

We construct a partition of unity for the Dirac field. For general properties of partition of unity for fermionic fields, refer to Ammari [1].

Let

$$c_{\tau,s}(f) = \begin{cases} b_s(f), & \tau = +, \\ d_s(f), & \tau = -. \end{cases}$$

Let $U_f : \mathcal{F}_f(L^2(\mathbf{R}_p^3; \mathbf{C}^4) \oplus L^2(\mathbf{R}_p^3; \mathbf{C}^4)) \rightarrow \mathcal{F}_{\text{Dirac}} \otimes \mathcal{F}_{\text{Dirac}}$ be an isometric operator which satisfy $U_f \Omega_D = \Omega_D \otimes \Omega_D$ and

$$\begin{aligned} & U_f c_{\tau_1, s_1}^\dagger \left(\begin{bmatrix} f_1 \\ g_1 \end{bmatrix} \right) \cdots c_{\tau_1, s_1}^\dagger \left(\begin{bmatrix} f_1 \\ g_n \end{bmatrix} \right) \Omega_D \\ &= \left(c_{\tau_1, s_1}^\dagger(f_1) \otimes \mathbb{1} + (-1)^{N_D} \otimes c_{\tau_1, s_1}^\dagger(g_1) \right) \cdots \left(c_{\tau_n, s_n}^\dagger(f_n) \otimes \mathbb{1} + (-1)^{N_D} \otimes c_{\tau_n, s_n}^\dagger(g_n) \right) \Omega_D \otimes \Omega_D. \end{aligned}$$

Here note that $(-1)^{N_D} \Psi = (-1)^n \Psi$ for the vector of the form $\Psi = c_{\tau_1, s_1}^\dagger(f_1) \cdots c_{\tau_n, s_n}^\dagger(f_n) \Omega_D$, $f_j \in L^2(\mathbf{R}^3)$, $j = 1, \dots, n$, $n \in \mathbf{N}$. Let $j_0, j_\infty \in C^\infty(\mathbf{R})$. We assume that $j_0 \geq 0$, $j_\infty \geq 0$, $j_0(\mathbf{x})^2 + j_\infty(\mathbf{x})^2 = 1$, $j_0(\mathbf{x}) = 1$ for $|\mathbf{x}| \leq 1$ and $j_0(\mathbf{x}) = 0$ for $|\mathbf{x}| \geq 2$. Let $j_{f,R} = \begin{bmatrix} j_{f,R}^0 \\ j_{f,R}^\infty \end{bmatrix}$ where $j_{f,R}^0 = j_0(\frac{-i\nabla_p}{R})$ and $j_{f,R}^\infty = j_\infty(\frac{-i\nabla_p}{R})$ with $\nabla_p = (\partial_{p^1}, \partial_{p^2}, \partial_{p^3})$.

Let $X_{f,R} : \mathcal{F}_{\text{Dir}} \rightarrow \mathcal{F}_{\text{Dir}} \otimes \mathcal{F}_{\text{Dir}}$ defined by

$$X_{f,R} = U_f \Gamma_f(j_{f,R}).$$

From (15)-(17), it holds that

$$X_{f,R} H_D = \left(H_D \otimes \mathbb{1} + \mathbb{1} \otimes H_D \right) X_{f,R} + U_f d\Gamma_f(j_{f,R}, \tilde{\text{ad}}_{\omega_M}(j_{f,R})), \quad (22)$$

$$X_{f,R} c_{\tau,s}(f) = (c_{\tau,s}(f) \otimes \mathbb{1}) X_{f,R} + X_{f,R} c_{\tau,s}((1 - j_{f,R}^0)f), \quad (23)$$

$$X_{f,R} c_{\tau,s}^\dagger(f) = (c_{\tau,s}^*(f) \otimes \mathbb{1}) X_{f,R} + (c_{\tau,s}^\dagger((j_{f,R}^0 - 1)f) \otimes \mathbb{1} + (-1)^{N_D} \otimes c_{\tau,s}^\dagger(j_{f,R}^\infty f)) X_{f,R}. \quad (24)$$

Lemma 3.1 Assume (A.1). Then,

- (i) $\|(X_{f,R} H_D - (H_D \otimes \mathbb{1} + \mathbb{1} \otimes H_D)) X_{f,R} (N_D + 1)^{-1}\| \leq \frac{c_f}{R},$
- (ii) $\|X_{f,R} \psi_l(\mathbf{x}) - (\psi_l(\mathbf{x}) \otimes \mathbb{1}) X_{f,R}\| \leq \delta_{f,R}^{1,l}(\mathbf{x}), \quad l = 1, \dots, 4,$
- (iii) $\|X_{f,R} \psi_l(\mathbf{x})^* - (\psi_l(\mathbf{x})^* \otimes \mathbb{1}) X_{f,R}\| \leq \delta_{f,R}^{2,l}(\mathbf{x}), \quad l = 1, \dots, 4.$

Here $c_f \geq 0$ is a constant, and $\delta_{f,R}^{i,l}(\mathbf{x}) \geq 0$, $l = 1, \dots, 4$, $i = 1, 2$, are error terms which satisfy $\sup_{\mathbf{x} \in \mathbf{R}^3} |\delta_{f,R}^{i,l}(\mathbf{x})| < \infty$ and $\lim_{R \rightarrow \infty} \delta_{f,R}^{i,l}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbf{R}^3$.

(Proof) (i) By (22), we have

$$\|(X_{f,R}H_D - (H_D \otimes \mathbb{1} + \mathbb{1} \otimes H_D)X_{f,R})(N_D + 1)^{-1}\| \leq \|d\Gamma_f(j_{f,R}, \tilde{\text{ad}}_{\omega_M}(j_{f,R}))(N_D + 1)^{-1}\|,$$

and (14) yields that

$$\|d\Gamma_f(j_{f,R}, \tilde{\text{ad}}_{\omega_M}(j_{f,R}))(N_D + 1)^{-1}\| \leq \sqrt{\|[\omega_M, j_{f,R}^0]\|_{B(L^2(\mathbf{R}^3))}^2 + \|[\omega_M, j_{f,R}^\infty]\|_{B(L^2(\mathbf{R}^3))}^2}$$

By pseudo-differential calculus (e.g., [13] ; Appendix A, [19] ; Section IV), it follows that

$\|[\omega_M, j_{f,R}^\sharp]\|_{B(L^2(\mathbf{R}^3))} \leq \frac{c_\sharp}{R}$, $\sharp = 0, \infty$, where $c_\sharp \geq 0$ are constants. Thus **(i)** is proven.

(ii) By the definition of $\psi_l(\mathbf{x}) = \sum_{s=\pm 1/2} (b_s(f_{s,\mathbf{x}}^l) + d_s^\dagger(g_{s,\mathbf{x}}^l))$, we have from (23) and (24) that

$$\begin{aligned} & X_{f,R} \psi_l(\mathbf{x}) - (\psi_l(\mathbf{x}) \otimes \mathbb{1})X_{f,R} \\ &= \sum_{s=\pm 1/2} \left(X_{f,R} b_s((1 - j_{f,R}^0)f_{s,\mathbf{x}}^l) + \left(d_s^\dagger((j_{f,R}^0 - 1)g_{s,\mathbf{x}}^l) \otimes \mathbb{1} + (-1)^{N_D} \otimes d_s^\dagger(j_{f,R}^\infty g_{s,\mathbf{x}}^l) \right) X_{f,R} \right). \end{aligned}$$

Then we have

$$\begin{aligned} & \|X_{f,R} \psi_l(\mathbf{x}) - (\psi_l(\mathbf{x}) \otimes \mathbb{1})X_{f,R}\| \\ & \leq \sum_{s=\pm 1/2} \left(\|b_s((1 - j_{f,R}^0)f_{s,\mathbf{x}}^l)\| + \|(d_s^\dagger((j_{f,R}^0 - 1)g_{s,\mathbf{x}}^l) \otimes \mathbb{1})X_{f,R}\| + \|(\mathbb{1} \otimes d_s^\dagger(j_{f,R}^\infty g_{s,\mathbf{x}}^l)X_{f,R})\| \right) \\ & \leq \sum_{s=\pm 1/2} \left(\|((1 - j_{f,R}^0)f_{s,\mathbf{x}}^l)\| + \|((j_{f,R}^0 - 1)g_{s,\mathbf{x}}^l)\| + \|j_{f,R}^\infty g_{s,\mathbf{x}}^l\| \right). \end{aligned}$$

Let $\delta_{f,R}^{1,l}(\mathbf{x}) = \sum_{s=\pm 1/2} \left(\|((1 - j_{f,R}^0)f_{s,\mathbf{x}}^l)\| + \|((j_{f,R}^0 - 1)g_{s,\mathbf{x}}^l)\| + \|j_{f,R}^\infty g_{s,\mathbf{x}}^l\| \right)$. We see that $\sup_{\mathbf{x} \in \mathbf{R}^3} |\delta_{f,R}^{1,l}(\mathbf{x})| \leq \sum_{s=\pm 1/2} (\|f_s^l\| + 2\|g_s^l\|)$ and $\lim_{R \rightarrow \infty} \delta_{f,R}^{1,l}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbf{R}$. Hence **(ii)** follows.

(iii) From the definition of $\psi_l(\mathbf{x})^* = \sum_{s=\pm 1/2} (b_s^\dagger(f_{s,\mathbf{x}}^l) + d_s(g_{s,\mathbf{x}}^l))$, (23) and (24) yield that

$$\begin{aligned} & X_{f,R} \psi_l(\mathbf{x})^* - (\psi_l(\mathbf{x})^* \otimes \mathbb{1})X_{f,R} \\ &= \sum_{s=\pm 1/2} \left(\left(b_s^\dagger((j_{f,R}^0 - 1)f_{s,\mathbf{x}}^l) \otimes \mathbb{1} + (-1)^{N_D} \otimes b_s^\dagger(j_{f,R}^\infty f_{s,\mathbf{x}}^l) \right) X_{f,R} + X_{f,R} d_s((1 - j_{f,R}^0)g_{s,\mathbf{x}}^l) \right). \end{aligned}$$

Then it follows that

$$\|X_{f,R} \psi_l(\mathbf{x})^* - (\psi_l(\mathbf{x})^* \otimes \mathbb{1})X_{f,R}\| \leq \delta_{f,R}^{2,l}(\mathbf{x}),$$

where $\delta_{f,R}^{2,l}(\mathbf{x}) = \sum_{s=\pm 1/2} \left(\|((j_{f,R}^0 - 1)f_{s,\mathbf{x}}^l)\| + \|j_{f,R}^\infty f_{s,\mathbf{x}}^l\| + \|((1 - j_{f,R}^0)g_{s,\mathbf{x}}^l)\| \right)$. It is seen that $\sup_{\mathbf{x} \in \mathbf{R}^3} |\delta_{f,R}^{2,l}(\mathbf{x})| \leq \sum_{s=\pm 1/2} (2\|f_s^l\| + \|g_s^l\|)$ and $\lim_{R \rightarrow \infty} \delta_{f,R}^{2,l}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbf{R}$. Thus we obtain **(iii)**. ■

Corollary 3.2 Assume (A.1). Then, for all $l, l' = 1, \dots, 4$,

$$\begin{aligned} \text{(i)} \quad & \|X_{f,R} \psi_l(\mathbf{x})^* \psi_{l'}(\mathbf{x}) - (\psi_l(\mathbf{x})^* \psi_{l'}(\mathbf{x}) \otimes \mathbb{1}) X_{f,R}\| \leq \delta_{f,R}^{3,l,l'}(\mathbf{x}), \\ \text{(ii)} \quad & \|X_{f,R} \psi_l(\mathbf{x})^* \psi_l(\mathbf{x}) \psi_{l'}(\mathbf{y})^* \psi_{l'}(\mathbf{y}) - (\psi_l(\mathbf{x})^* \psi_l(\mathbf{x}) \psi_{l'}(\mathbf{y})^* \psi_{l'}(\mathbf{y}) \otimes \mathbb{1}) X_{f,R}\| \leq \delta_{f,R}^{4,l,l'}(\mathbf{x}, \mathbf{y}). \end{aligned}$$

Here $\delta_{f,R}^{3,l,l'}(\mathbf{x}) \geq 0$ satisfies $\sup_{\mathbf{x} \in \mathbf{R}^3} |\delta_{f,R}^{3,l,l'}(\mathbf{x})| < \infty$ and $\lim_{R \rightarrow \infty} \delta_{f,R}^{3,l,l'}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbf{R}^3$, and $\delta_{f,R}^{4,l,l'}(\mathbf{x}, \mathbf{y}) \geq 0$ satisfies $\sup_{(\mathbf{x}, \mathbf{y}) \in \mathbf{R}^3 \times \mathbf{R}^3} |\delta_{f,R}^{4,l,l'}(\mathbf{x}, \mathbf{y})| < \infty$ and $\lim_{R \rightarrow \infty} \delta_{f,R}^{4,l,l'}(\mathbf{x}, \mathbf{y}) = 0$ for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3$.

(Proof) (i) By Lemma 3.1 (ii) and (iii), it is seen that

$$\begin{aligned} & \|X_{f,R} \psi_l(\mathbf{x})^* \psi_{l'}(\mathbf{x}) - (\psi_l(\mathbf{x})^* \psi_{l'}(\mathbf{x}) \otimes \mathbb{1}) X_{f,R}\| \\ & \leq \|(X_{f,R} \psi_l(\mathbf{x})^* - ((\psi_l(\mathbf{x})^* \otimes \mathbb{1}) X_{f,R})) \psi_{l'}(\mathbf{x})\| \\ & \quad + \|(\psi_l(\mathbf{x})^* \otimes \mathbb{1}) (X_{f,R} \psi_{l'}(\mathbf{x}) - (\psi_{l'}(\mathbf{x}) \otimes \mathbb{1}) X_{f,R})\| \\ & \leq \delta_{f,R}^{2,l} \|\psi_{l'}(\mathbf{x})\| + \delta_{f,R}^{1,l'} \|\psi_l(\mathbf{x})^*\|. \end{aligned}$$

Note that $\|\psi_{l'}(\mathbf{x})\| \leq c_D^{l'}$ and $\|\psi_l(\mathbf{x})^*\| \leq c_D^l$. Hence (i) is obtained. Similarly, we can prove (ii) by using (i). ■

3.3 Partition of Unity for Radiation Field

Let $U_b : \mathcal{F}_b(L^2(\mathbf{R}_k^3 \times \{1, 2\}) \oplus L^2(\mathbf{R}_k^3 \times \{1, 2\})) \rightarrow \mathcal{F}_{\text{rad}} \otimes \mathcal{F}_{\text{rad}}$ an isometric operator satisfying $U_b \Omega_{\text{rad}} = \Omega_{\text{rad}} \otimes \Omega_{\text{rad}}$ and

$$\begin{aligned} & U_b a_{r_1}^\dagger \left(\begin{bmatrix} f_1 \\ g_1 \end{bmatrix} \right) \cdots a_{r_1}^\dagger \left(\begin{bmatrix} f_1 \\ g_n \end{bmatrix} \right) \Omega_{\text{rad}} \\ & = \left(a_{r_1}^\dagger(f_1) \otimes \mathbb{1} + \mathbb{1} \otimes a_{r_1}^\dagger(g_1) \right) \cdots \left(a_{r_n}^\dagger(f_n) \otimes \mathbb{1} + \mathbb{1} \otimes a_{r_n}^\dagger(g_n) \right) \Omega_{\text{rad}} \otimes \Omega_{\text{rad}}. \end{aligned}$$

Let $j_0, j_\infty \in C^\infty(\mathbf{R})$. We suppose that $j_0 \geq 0$, $j_\infty \geq 0$, $j_0^2 + j_\infty^2 = 1$, $j_0(\mathbf{y}) = 1$ if $|\mathbf{y}| \leq 1$ and $j_0(\mathbf{y}) = 0$ if $|\mathbf{y}| \geq 2$. We set $j_{b,R} = \begin{bmatrix} j_{b,R}^0 \\ j_{b,R}^\infty \end{bmatrix}$ where $j_{b,R}^0 = j_0(\frac{-i\nabla_{\mathbf{k}}}{R})$ and $j_{b,R}^\infty = j_\infty(\frac{-i\nabla_{\mathbf{k}}}{R})$ with $\nabla_{\mathbf{k}} = (\partial_{k^1}, \partial_{k^2}, \partial_{k^3})$.

Let $Y_{b,R} : \mathcal{F}_{\text{rad}} \rightarrow \mathcal{F}_{\text{rad}} \otimes \mathcal{F}_{\text{rad}}$ defined by

$$Y_{b,R} = U_b \Gamma_b(j_{b,R}).$$

From (19) - (21), it follows that

$$Y_{b,R} H_{\text{rad},m} = \left(H_{\text{rad},m} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\text{rad},m} \right) Y_{b,R} - U_b d\Gamma_b(j_{b,R}, \tilde{\text{ad}}_{\omega_m}(j_{b,R})), \quad (25)$$

$$Y_{b,R} a_r(h) = (a_r(h) \otimes \mathbb{1}) Y_{b,R} + Y_{b,R} a_r((1 - j_{b,R}^0)h), \quad (26)$$

$$Y_{b,R} a_r^\dagger(h) = (a_r^*(h) \otimes \mathbb{1}) Y_{b,R} + (a_r^\dagger((j_{b,R}^0 - 1)h) \otimes \mathbb{1} + \mathbb{1} \otimes a_r^\dagger(j_{b,R}^\infty h)) Y_{b,R}. \quad (27)$$

Lemma 3.3 Assume (A.2). Then

$$\begin{aligned} \text{(i)} \quad & \left\| \left(Y_{b,R} H_{\text{rad},m} - (H_{\text{rad},m} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\text{rad},m}) Y_{b,R} \right) (N_{\text{rad}} + 1)^{-1} \right\| \leq \frac{c_b}{R}, \\ \text{(ii)} \quad & \left\| \left(Y_{b,R} A_j(\mathbf{x}) - (A_j(\mathbf{x}) \otimes \mathbb{1}) Y_{b,R} \right) (N_{\text{rad}} + 1)^{-1/2} \right\| \leq \delta_{b,R}^j(\mathbf{x}). \end{aligned}$$

Here $c_b \geq 0$ is a constant and $\delta_{b,R}^j(\mathbf{x}) \geq 0$, $j = 1, 2, 3$, are error terms which satisfy $\sup_{\mathbf{x} \in \mathbf{R}^3} |\delta_{b,R}^j(\mathbf{x})| < \infty$ and $\lim_{R \rightarrow \infty} \delta_{b,R}^j(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbf{R}^3$.

(Proof) (i) It is proven in a similar way to Lemma 3.1 (i).

(ii) By the definition of $A_j(\mathbf{x}) = \sum_{r=1,2} \left(a_r(h_{r,\mathbf{x}}^j) + a_r^\dagger(h_{r,\mathbf{x}}^j) \right)$, it follows from (26) and (27) that

$$\begin{aligned} & Y_{b,R} A_j(\mathbf{x}) - (A_j(\mathbf{x}) \otimes \mathbb{1}) Y_{b,R} \\ &= \sum_{r=1,2} \left(Y_{b,R} a_r((1 - j_{b,R}^0) h_{r,\mathbf{x}}^j) + (a_r^\dagger((j_{b,R}^0 - 1) h_{r,\mathbf{x}}^j) \otimes \mathbb{1} + \mathbb{1} \otimes a_r^\dagger(j_{b,R}^\infty h_{r,\mathbf{x}}^j)) Y_{b,R} \right). \end{aligned}$$

Since $\|a_r(h)(N_{\text{rad}} + 1)^{-1/2}\| \leq \|h\|$ and $\|a_r^\dagger(h)(N_{\text{rad}} + 1)^{-1/2}\| \leq 2\|h\|$, we have

$$\begin{aligned} & \left\| (Y_{b,R} A_j(\mathbf{x}) - (A_j(\mathbf{x}) \otimes \mathbb{1}) Y_{b,R}) (N_{\text{rad}} + 1)^{-1/2} \right\| \\ & \leq \sum_{r=1,2} \left(\|(a_r(1 - j_{b,R}^0) h_{r,\mathbf{x}}^j)(N_{\text{rad}} + 1)^{-1/2}\| \right. \\ & \quad + \|(a_r^\dagger((j_{b,R}^0 - 1) h_{r,\mathbf{x}}^j)(N_{\text{rad}} + 1)^{-1/2} \otimes \mathbb{1})(N_{\text{rad}} + 1)^{1/2} \otimes \mathbb{1}) Y_{b,R} (N_{\text{rad}} + 1)^{-1/2}\| \\ & \quad \left. + \|(\mathbb{1} \otimes a_r^\dagger(j_{b,R}^\infty h_{r,\mathbf{x}}^j)(N_{\text{rad}} + 1)^{-1/2})(\mathbb{1} \otimes (N_{\text{rad}} + 1)^{1/2}) Y_{b,R} (N_{\text{rad}} + 1)^{-1/2}\| \right) \\ & \leq \sum_{r=1,2} \left(\|(1 - j_{b,R}^0) h_{r,\mathbf{x}}^j\| + 2\|(j_{b,R}^0 - 1) h_{r,\mathbf{x}}^j\| + 2\|j_{b,R}^\infty h_{r,\mathbf{x}}^j\| \right). \end{aligned}$$

Let $\delta_{b,R}^j(\mathbf{x}) = \sum_{r=1,2} \left(3\|((1 - j_{b,R}^0) h_{r,\mathbf{x}}^j)\| + 2\|j_{b,R}^\infty h_{r,\mathbf{x}}^j\| \right)$, $j = 1, 2, 3$. We see that $\sup_{\mathbf{x} \in \mathbf{R}^3} |\delta_{b,R}^j(\mathbf{x})| \leq 5(\|h_1^j\| + \|h_2^j\|)$ and $\lim_{R \rightarrow \infty} \delta_{b,R}^j(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbf{R}$. Thus we obtain the proof. ■

3.4 Existence of Ground State of H_m

We recall that the massive Hamiltonian is defined by

$$H_m = H_D \otimes \mathbb{1}_{\text{rad}} + \mathbb{1}_D \otimes H_{\text{rad},m} + \kappa_I H_I + \kappa_{II} H_{II}.$$

Throughout this subsection, we do not omit the subscripts of the identities $\mathbb{1}_D$ and $\mathbb{1}_{\text{rad}}$.

Since $\frac{1}{\omega_m(\mathbf{k})^\lambda} \leq \frac{1}{\omega(\mathbf{k})^\lambda}$, $\lambda > 0$, it holds that

$$\|A_j(\mathbf{x})(H_{\text{rad},m} + 1)^{-1/2}\| \leq \sum_{r=1,2} \left(2\left\| \frac{\chi_{\text{rad}} e_r^j}{\omega_m} \right\| + \left\| \frac{\chi_{\text{rad}} e_r^j}{\sqrt{\omega_m}} \right\| \right) \leq c_{\text{rad}}^j. \quad (28)$$

Then, we have

$$\|H_I \Psi\| \leq c_I \|\mathbb{1} \otimes (H_{\text{rad},m} + 1)^{1/2} \Psi\|, \quad (29)$$

and it holds that for all $\varepsilon > 0$,

$$\|H_I \Psi\| \leq c_I \varepsilon \|H_{0,m} \Psi\| + c_I \left(\frac{1}{2\varepsilon} + 1 \right) \|\Psi\|. \quad (30)$$

From (30) and $\|H_{II}\| < \infty$, it is proven that H_m is self-adjoint and essentially self adjoint on any core of $H_{0,m}$.

Theorem 3.4 (Existence of a Ground State of H_m)

Suppose (A.1) - (A.3). Let $m < M$. Then H_m has purely discrete spectrum in $[E_0(H_m), E_0(H_m) + m)$. In particular, H_m has a ground state.

To prove Theorem 3.4, we need some preparations. We define $\tilde{X}_{f,R} : \mathcal{F}_{\text{QED}} \rightarrow \mathcal{F}_{\text{Dir}} \otimes \mathcal{F}_{\text{Dir}} \otimes \mathcal{F}_{\text{rad}}$ by

$$\tilde{X}_{f,R} = X_{f,R} \otimes \mathbb{1}_{\text{rad}}.$$

We introduce Hamiltonian $\tilde{H}_m : \mathcal{F}_{\text{Dir}} \otimes \mathcal{F}_{\text{Dir}} \otimes \mathcal{F}_{\text{rad}} \rightarrow \mathcal{F}_{\text{Dir}} \otimes \mathcal{F}_{\text{Dir}} \otimes \mathcal{F}_{\text{rad}}$ defined by

$$\tilde{H}_m = \tilde{H}_D \otimes \mathbb{1}_{\text{rad}} + \mathbb{1}_D \otimes \tilde{H}_{\text{rad}} + \kappa_I \tilde{H}_I + \kappa_{II} \tilde{H}_{II},$$

where $\tilde{H}_D = H_D \otimes \mathbb{1}_D$, $\tilde{H}_{\text{rad}} = \mathbb{1}_D \otimes H_{\text{rad},m}$ and

$$\begin{aligned} \tilde{H}_I &= \sum_{j=1}^3 \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) (\tilde{\psi}^\dagger(\mathbf{x}) \tilde{\alpha}^j \tilde{\psi}(\mathbf{x}) \otimes A_j(\mathbf{x})) d\mathbf{x}, \\ \tilde{H}_{II} &= \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{II}(\mathbf{x}) \chi_{II}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} (\tilde{\psi}^\dagger(\mathbf{x}) \tilde{\psi}(\mathbf{x}) \tilde{\psi}^\dagger(\mathbf{y}) \tilde{\psi}(\mathbf{y}) \otimes \mathbb{1}_{\text{rad}}) d\mathbf{x} d\mathbf{y}. \end{aligned}$$

with $\tilde{\psi}(\mathbf{x}) = \psi(\mathbf{x}) \otimes \mathbb{1}_D$ and $\tilde{\alpha}^j = \alpha^j \otimes \mathbb{1}_D$, $j = 1, \dots, 3$.

Proposition 3.5 Assume (A.1) - (A.3). Let $\Psi \in \mathcal{D}(H_m)$. Then, it holds that

- (i) $\|(\tilde{X}_{f,R}(H_D \otimes \mathbb{1}_{\text{rad}}) - (\tilde{H}_D \otimes \mathbb{1}_{\text{rad}} + \mathbb{1}_D \otimes H_D \otimes \mathbb{1}_{\text{rad}}) \tilde{X}_{f,R}) \Psi\| \leq \frac{c_f}{R} (\|(N_D \otimes \mathbb{1}_{\text{rad}}) \Psi\| + \|\Psi\|),$
- (ii) $\|(\tilde{X}_{f,R} H_I - \tilde{H}_I \tilde{X}_{f,R}) \Psi\| \leq \delta_{f,I}(R) (\|(\mathbb{1}_D \otimes N_{\text{rad}}^{1/2}) \Psi\| + \|\Psi\|),$
- (iii) $\|(\tilde{X}_{f,R} H_{II} - \tilde{H}_{II} \tilde{X}_{f,R}) \Psi\| \leq \delta_{f,II}(R) \|\Psi\|.$

Here $c_f \geq 0$ is the constant in Lemma 3.1(i), and $\delta_{f,I}(R) \geq 0$ and $\delta_{f,II}(R) \geq 0$ are error terms satisfying that $\lim_{R \rightarrow \infty} \delta_{f,I}(R) = 0$ and $\lim_{R \rightarrow \infty} \delta_{f,II}(R) = 0$, respectively.

(Proof)

(i) It directly follows from Lemma 3.1 (i).

(ii) Let $\Psi \in \mathcal{D}(H_m)$ and $\tilde{\Phi} \in \mathcal{F}_{\text{Dir}} \otimes \mathcal{F}_{\text{Dir}} \otimes \mathcal{F}_{\text{rad}}$ with $\|\tilde{\Phi}\| = 1$. Then,

$$\begin{aligned} & (\tilde{\Phi}, (\tilde{X}_{f,R} H_I - \tilde{H}_I \tilde{X}_{f,R}) \Psi) \\ &= \sum_{j=1}^3 \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) (\Phi, ((X_{f,R} \psi^\dagger(\mathbf{x})(\mathbf{x}) \alpha^j \psi(\mathbf{x}) - \tilde{\psi}^\dagger(\mathbf{x}) \tilde{\alpha}^j \tilde{\psi}(\mathbf{x}) X_{f,R}) \otimes A_j(\mathbf{x})) \Psi) d\mathbf{x}. \end{aligned}$$

Then we have

$$\begin{aligned} & |(\tilde{\Phi}, (\tilde{X}_{f,R} H_I - \tilde{H}_I \tilde{X}_{f,R}) \Psi)| \\ &\leq \sum_{j=1}^3 \int_{\mathbf{R}^3} |\chi_I(\mathbf{x})| \|X_{f,R} \psi^\dagger(\mathbf{x}) \alpha^j \psi(\mathbf{x}) - \tilde{\psi}^\dagger(\mathbf{x}) \tilde{\alpha}^j \tilde{\psi}(\mathbf{x}) X_{f,R}\| \|(\mathbb{1}_D \otimes A_j(\mathbf{x})) \Psi\| d\mathbf{x} \\ &\leq \sum_{j=1}^3 \sum_{l,l'=1}^4 |\alpha_{l,l'}^j| \int_{\mathbf{R}^3} |\chi_I(\mathbf{x})| \|(X_{f,R} \psi_l(\mathbf{x})^* \psi_{l'}(\mathbf{x}) - \tilde{\psi}_l(\mathbf{x})^* \tilde{\psi}_{l'}(\mathbf{x}) X_{f,R})\| \|(\mathbb{1}_D \otimes A_j(\mathbf{x})) \Psi\| d\mathbf{x}. \end{aligned}$$

By Corollary 3.2 (i), we have $\|(X_{f,R} \psi_l(\mathbf{x})^* \psi_{l'}(\mathbf{x}) - \tilde{\psi}_l(\mathbf{x})^* \tilde{\psi}_{l'}(\mathbf{x}) X_{f,R})\| \leq \delta_{f,R}^{3,l,l'}(\mathbf{x})$. We also see that $\|A_j(\mathbf{x})(N_{\text{rad}} + 1)^{-1/2}\| \leq 3 \sum_{r=1,2} \|h_r^j\|$. Then it follows that

$$\begin{aligned} & |(\tilde{\Phi}, (\tilde{X}_{f,R} H_I - \tilde{H}_I \tilde{X}_{f,R}) \Psi)| \\ &\leq \sum_{r=1,2} \sum_{j=1}^3 \sum_{l,l'=1}^4 |\alpha_{l,l'}^j| \|h_r^j\| \left(\int_{\mathbf{R}^3} |\chi_I(\mathbf{x})| \delta_{f,R}^{3,l,l'}(\mathbf{x}) d\mathbf{x} \right) \|(\mathbb{1}_D \otimes (N_{\text{rad}} + 1)^{1/2}) \Psi\|. \quad (31) \end{aligned}$$

Since (31) holds for all $\tilde{\Phi} \in \mathcal{F}_{\text{Dir}} \otimes \mathcal{F}_{\text{Dirac}} \otimes \mathcal{F}_{\text{rad}}$ with $\|\tilde{\Phi}\| = 1$, it follows that

$$\|(\tilde{X}_{f,R} H_I - \tilde{H}_I \tilde{X}_{f,R}) \Psi\| \leq \delta_{f,I}(R) \|(\mathbb{1}_D \otimes (N_{\text{rad}} + 1)^{1/2}) \Psi\|, \quad (32)$$

where $\delta_{f,I}(R) = 3 \sum_{r=1,2} \sum_{j=1}^3 \sum_{l,l'=1}^4 |\alpha_{l,l'}^j| \|h_r^j\| \|\chi_I \delta_{f,R}^{3,l,l'}\|_{L^1}$. We see that $\lim_{R \rightarrow \infty} \delta_{f,I}(R) = 0$, and hence (ii) follows.

(iii) Let $\Psi \in \mathcal{D}(H_m)$. We set $Q_l(\mathbf{x}) = \psi_l(\mathbf{x})^* \psi_l(\mathbf{x})$. Then for all $\tilde{\Phi} \in \mathcal{F}_{\text{Dir}} \otimes \mathcal{F}_{\text{Dir}} \otimes \mathcal{F}_{\text{rad}}$ with $\|\tilde{\Phi}\| = 1$,

$$\begin{aligned} & (\tilde{\Phi}, (\tilde{X}_{f,R} H_{\text{II}} - \tilde{H}_{\text{II}} \tilde{X}_{f,R}) \Psi) \\ &= \sum_{l,l'=1}^4 \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{\text{II}}(\mathbf{x}) \chi_{\text{II}}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} (\tilde{\Phi}, ((X_{f,R} Q_l(\mathbf{x}) Q_{l'}(\mathbf{y}) - (Q_l(\mathbf{x}) Q_{l'}(\mathbf{y}) \otimes \mathbb{1}_D) X_{f,R}) \otimes \mathbb{1}_{\text{rad}}) \Psi) d\mathbf{x} d\mathbf{y}. \end{aligned}$$

Then we have

$$\begin{aligned} & |(\tilde{\Phi}, (\tilde{X}_{f,R} H_{\text{II}} - \tilde{H}_{\text{II}} \tilde{X}_{f,R}) \Psi)| \\ &\leq \sum_{l,l'=1}^4 \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{|\chi_{\text{II}}(\mathbf{x}) \chi_{\text{II}}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|} \|X_{f,R} Q_l(\mathbf{x}) Q_{l'}(\mathbf{y}) - (Q_l(\mathbf{x}) Q_{l'}(\mathbf{y}) \otimes \mathbb{1}_D) X_{f,R}\| \Psi\| d\mathbf{x} d\mathbf{y}. \end{aligned}$$

From Corollary 3.2 (ii), it holds that $\|X_{f,R}Q_l(\mathbf{x})Q_{l'}(\mathbf{y}) - (Q_l(\mathbf{x})Q_{l'}(\mathbf{y}) \otimes \mathbb{1}_D)X_{f,R}\| \leq \delta_{f,R}^{4,l,l'}(\mathbf{x},\mathbf{y})$. Then we have

$$|(\tilde{\Phi}, (\tilde{X}_{f,R}H_{\Pi} - \tilde{H}_{\Pi}\tilde{X}_{f,R})\Psi)| \leq \left(\sum_{l,l'=1}^4 \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{|\chi_{\Pi}(\mathbf{x})\chi_{\Pi}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|} \delta_{f,R}^{4,l,l'}(\mathbf{x},\mathbf{y}) d\mathbf{x}d\mathbf{y} \right) \|\Psi\|.$$

This implies that

$$\|(\tilde{X}_{f,R}H_{\Pi} - \tilde{H}_{\Pi}\tilde{X}_{f,R})\Psi\| \leq \delta_{f,\Pi}(R) \|\Psi\|,$$

where $\delta_{f,\Pi}(R) = \sum_{l,l'=1}^4 \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{|\chi_{\Pi}(\mathbf{x})\chi_{\Pi}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|} \delta_{f,R}^{4,l,l'}(\mathbf{x},\mathbf{y}) d\mathbf{x}d\mathbf{y}$. We see that $\lim_{R \rightarrow \infty} \delta_{f,\Pi}(R) = 0$, and thus the proof is obtained. ■

We define $\tilde{Y}_{b,R} : \mathcal{F}_{\text{QED}} \rightarrow \mathcal{F}_{\text{Dir}} \otimes \mathcal{F}_{\text{rad}} \otimes \mathcal{F}_{\text{rad}}$ by

$$\tilde{Y}_{b,R} = \mathbb{1}_D \otimes Y_{b,R}.$$

Proposition 3.6 Assume (A.1) - (A.3). Then it holds that for all $\Psi \in \mathcal{D}(H_m)$,

$$\begin{aligned} \text{(i)} \quad & \|(\tilde{Y}_{b,R}(\mathbb{1}_D \otimes H_{\text{rad},m}) - (\mathbb{1}_D \otimes H_{\text{rad},m} \otimes \mathbb{1}_{\text{rad}} + \mathbb{1}_{\text{QED}} \otimes H_{\text{rad},m})\tilde{Y}_{b,R})\Psi\| \\ & \leq \frac{c_b}{R} (\|(\mathbb{1}_D \otimes N_{\text{rad}})\Psi\| + \|\Psi\|), \\ \text{(ii)} \quad & \|(\tilde{Y}_{b,R}H_I - (H_I \otimes \mathbb{1}_{\text{rad}})\tilde{Y}_{b,R})\Psi\| \leq \delta_{b,I}(R) (\|(\mathbb{1}_D \otimes N_{\text{rad}}^{1/2})\Psi\| + \|\Psi\|), \end{aligned}$$

where $c_b \geq 0$ and $\delta_{b,I}(R) \geq 0$ satisfying $\lim_{R \rightarrow \infty} \delta_{b,I}(R) = 0$.

(Proof) (i) It immediately follows from Lemma 3.3 (i).

(ii) Let $\Psi \in \mathcal{D}(H_m)$ and $\tilde{\Xi} \in \mathcal{F}_{\text{Dir}} \otimes \mathcal{F}_{\text{rad}} \otimes \mathcal{F}_{\text{rad}}$ with $\|\tilde{\Xi}\| = 1$. We see that

$$\begin{aligned} & (\tilde{\Xi}, (\tilde{Y}_{b,R}H_I - (H_I \otimes \mathbb{1}_{\text{rad}})\tilde{Y}_{b,R})\Psi) \\ & = \sum_{j=1}^3 \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) (\tilde{\Xi}, (\psi^\dagger(\mathbf{x})\alpha^j\psi(\mathbf{x})) \otimes (Y_{b,R}A_j(\mathbf{x}) - (A_j(\mathbf{x}) \otimes \mathbb{1}_{\text{rad}})Y_{b,R}))\Psi) d\mathbf{x}, \end{aligned}$$

Then,

$$\begin{aligned} & |(\tilde{\Xi}, (\tilde{Y}_{b,R}H_I - (H_I \otimes \mathbb{1}_{\text{rad}})\tilde{Y}_{b,R})\Psi)| \\ & \leq \sum_{j=1}^3 \int_{\mathbf{R}^3} |\chi_I(\mathbf{x})| \|(\psi^\dagger(\mathbf{x})\alpha^j\psi(\mathbf{x})) \otimes (Y_{b,R}A_j - (A_j(\mathbf{x}) \otimes \mathbb{1}_{\text{rad}})Y_{b,R}))\Psi\| d\mathbf{x} \\ & \leq \left(\sum_{j=1}^3 \sum_{l,l'=1}^4 |\alpha_{l,l'}^j| c_D^l c_D^{l'} \right) \int_{\mathbf{R}^3} |\chi_I(\mathbf{x})| \|(\mathbb{1}_D \otimes (Y_{b,R}A_j - (A_j(\mathbf{x}) \otimes \mathbb{1}_{\text{rad}})Y_{b,R}))\Psi\| d\mathbf{x}. \end{aligned}$$

From Lemma 3.3 (ii), it holds that

$$\|(\mathbb{1}_D \otimes (Y_{b,R}A_j - (A_j(\mathbf{x}) \otimes \mathbb{1}_{\text{rad}})Y_{b,R}))\Psi\| \leq \delta_{b,R}^j(\mathbf{x}) \|(\mathbb{1}_D \otimes (N_{\text{rad}} + 1)^{1/2})\Psi\|,$$

where $\delta_{b,R}^j(\mathbf{x}) \geq 0$ is the error term, and hence,

$$|(\tilde{\Xi}, (\tilde{Y}_{b,R} H_I - (H_I \otimes \mathbb{1}_{\text{rad}}) \tilde{Y}_{b,R}) \Psi)| \leq \delta_{b,I}(R) \|(\mathbb{1}_D \otimes (N_{\text{rad}} + 1)^{1/2}) \Psi\|, \quad (33)$$

where $\delta_{b,I}(R) = \sum_{l,l'=1}^4 |\alpha_{l,l'}^j| c_D^l c_D^{l'} \int_{\mathbf{R}^3} |\chi_I(\mathbf{x})| \delta_b^j(\mathbf{x}) dx$. Since (35) holds for all $\tilde{\Xi} \in \mathcal{F}_{\text{Dir}} \otimes \mathcal{F}_{\text{rad}} \otimes \mathcal{F}_{\text{rad}}$ with $\|\tilde{\Xi}\| = 1$, we have

$$\|(\tilde{Y}_{b,R} H_I - (H_I \otimes \mathbb{1}_{\text{rad}}) \tilde{Y}_{b,R}) \Psi\| \leq \delta_{b,I}(R) \|(\mathbb{1}_D \otimes (N_{\text{rad}} + 1)^{1/2}) \Psi\|.$$

Since $\lim_{R \rightarrow \infty} \delta_{b,I}(R) = 0$, the proof is obtained. ■

Here we introduce a new norm defined by

$$\|\Psi\|_{\lambda, \lambda'} = \|(N_D^{\lambda/2} \otimes \mathbb{1}_{\text{rad}}) \Psi\| + \|(\mathbb{1}_D \otimes N_{\text{rad}}^{\lambda'/2}) \Psi\| + \|\Psi\|, \quad \Psi \in \mathcal{D}(N_D^{\lambda/2} \otimes N_{\text{rad}}^{\lambda'/2}).$$

From Proposition 3.5 and Proposition 3.6, the next corollary follows.

Corollary 3.7 Assume (A.1) - (A.3). Then for all $\Psi \in \mathcal{D}(H_m)$,

- (i) $\|(\tilde{X}_{f,R} H_m - (\tilde{H}_m + \mathbb{1}_D \otimes H_D \otimes \mathbb{1}_{\text{rad}}) \tilde{X}_{f,R}) \Psi\| \leq \delta_f(R) \|\Psi\|_{2,1},$
- (ii) $\|(\tilde{Y}_{b,R} H_m - (H_m \otimes \mathbb{1}_{\text{rad}} + \mathbb{1}_D \otimes \mathbb{1}_{\text{rad}} \otimes H_{\text{rad},m}) \tilde{Y}_{b,R}) \Psi\| \leq \delta_b(R) \|\Psi\|_{0,2}.$

Here $\delta_f(R) \geq 0$ and $\delta_b(R) \geq 0$ are error terms which satisfy that $\lim_{R \rightarrow \infty} \delta_f(R) = 0$ and $\lim_{R \rightarrow \infty} \delta_b(R) = 0$, respectively.

Lemma 3.8 Assume (A.1) - (A.3). Let $q_{f,R} = (j_{f,R}^0)^2$ and $q_{b,R} = (j_{b,R}^0)^2$. Then, for all $\Psi \in \mathcal{D}(H_m)$ with $\|\Psi\| = 1$,

$$\begin{aligned} (\Psi, H_m \Psi) &\geq E_0(H_m) + m + (M - m) (\Psi, (\mathbb{1}_D \otimes \Gamma_b(q_{b,R})) \Psi) \\ &\quad - M (\Psi, (\Gamma_f(q_{f,R}) \otimes \Gamma_b(q_{b,R})) \Psi) + (\delta_f(R) \|\Psi\|_{2,1} + \delta_b(R) \|\Psi\|_{0,2}). \end{aligned}$$

(Proof) Let $\Psi \in \mathcal{D}(H_m)$ with $\|\Psi\| = 1$. By Lemma Corollary 3.7 (ii),

$$\begin{aligned} (\Psi, H_m \Psi) &= (\Psi, \tilde{Y}_{b,R}^* \tilde{Y}_{b,R} H_m \Psi) \\ &\geq (\Psi, \tilde{Y}_{b,R}^* (H_m \otimes \mathbb{1}_{\text{rad}}) \tilde{Y}_{b,R} \Psi) + (\Psi, \tilde{Y}_{b,R}^* (\mathbb{1}_D \otimes \mathbb{1}_{\text{rad}} \otimes H_{\text{rad},m}) \tilde{Y}_{b,R} \Psi) - \delta_b(R) \|\Psi\|_{0,2}. \end{aligned}$$

We see that $H_{\text{rad},m} \geq m(\mathbb{1}_{\text{rad}} - P_{\text{rad}})$ with $P_{\text{rad}} = E_{N_{\text{rad}}}(\{0\})$ where $E_X(J)$ denotes the spectral projection on a Borel set $J \in \mathcal{B}(\mathbf{R})$ for a self-adjoint operator X . Then

$$\begin{aligned} (\Psi, H_m \Psi) &\geq (\Psi, \tilde{Y}_{\text{rad},R}^* (H_m \otimes \mathbb{1}_{\text{rad}}) \tilde{Y}_{\text{rad},R} \Psi) + m \\ &\quad - m (\Psi, \tilde{Y}_{\text{rad},R}^* (\mathbb{1}_D \otimes \mathbb{1}_{\text{rad}} \otimes P_{\text{rad}}) \tilde{Y}_{\text{rad},R} \Psi) - \delta_b(R) \|\Psi\|_{0,2} \\ &\geq (\Psi, \tilde{Y}_{b,R}^* (H_m \otimes \mathbb{1}_{\text{rad}}) \tilde{Y}_{b,R} \Psi) + m - m (\Psi, (\mathbb{1}_D \otimes \Gamma_b(q_{b,R})) \Psi) - \delta_b(R) \|\Psi\|_{0,2}. \quad (34) \end{aligned}$$

Here we used $Y_{b,R}^*(\mathbb{1}_{\text{rad}} \otimes P_{\text{rad}})Y_{b,R} = \Gamma_b(q_{b,R})$ in the last line. We evaluate the first term in the right hand side of (34). Let $\tilde{X}_{f,R} = \tilde{X}_{f,R} \otimes \mathbb{1}_{\text{rad}}$. By Corollary 3.7 (i),

$$\begin{aligned}
& (\Psi, \tilde{Y}_{b,R}^*(H_m \otimes \mathbb{1}_{\text{rad}}) \tilde{Y}_{b,R} \Psi) \\
&= (\Psi, \tilde{Y}_{b,R}^*((\tilde{X}_{f,R}^* \tilde{X}_{f,R} H_m) \otimes \mathbb{1}_{\text{rad}}) \tilde{Y}_{b,R} \Psi) \\
&\geq (\Psi, \tilde{Y}_{\text{rad},R}^* \tilde{X}_{f,R}^* (\tilde{H}_m \otimes \mathbb{1}_{\text{rad}}) \tilde{X}_{f,R} \tilde{Y}_{b,R} \Psi) \\
&\quad + (\Psi, \tilde{Y}_{b,R}^* \tilde{X}_{f,R}^* (\mathbb{1}_D \otimes H_D \otimes \mathbb{1}_{\text{rad}}) \tilde{X}_{b,R} \tilde{Y}_{b,R} \Psi) - \delta_f(R) \|\tilde{Y}_{b,R} \Psi\|_{2,1}^{\sim}, \tag{35}
\end{aligned}$$

where we set

$$\|\tilde{\Phi}\|_{\lambda,\lambda'}^{\sim} = \|(N_D^{\lambda/2} \otimes \mathbb{1}_{\text{rad}} \otimes \mathbb{1}_{\text{rad}}) \tilde{\Phi}\| + \|(\mathbb{1}_D \otimes N_{\text{rad}}^{\lambda'/2} \otimes \mathbb{1}_{\text{rad}}) \tilde{\Phi}\| + \|\tilde{\Phi}\|,$$

for $\tilde{\Phi} \in \mathcal{D}(N_D^{\lambda/2} \otimes \mathbb{1}_{\text{rad}} \otimes \mathbb{1}_{\text{rad}}) \cap \mathcal{D}(\mathbb{1}_D \otimes N_{\text{rad}}^{\lambda'/2} \otimes \mathbb{1}_{\text{rad}})$. We see that

$$\begin{aligned}
\|\tilde{Y}_{b,R} \Psi\|_{2,1}^{\sim} &= \|\tilde{Y}_{b,R}(N_D \otimes \mathbb{1}_{\text{rad}}) \Psi\| + \|(\mathbb{1}_D \otimes N_{\text{rad}}^{1/2} \otimes \mathbb{1}_{\text{rad}}) \tilde{Y}_{b,R} \Psi\| + \|\tilde{Y}_{b,R} \Psi\| \\
&\leq \|\tilde{Y}_{b,R}(N_D \otimes \mathbb{1}_{\text{rad}}) \Psi\| + \|\tilde{Y}_{b,R}(\mathbb{1}_D \otimes N_{\text{rad}}^{1/2}) \Psi\| + \|\tilde{Y}_{b,R} \Psi\| = \|\Psi\|_{2,1},
\end{aligned}$$

and $H_D \geq M(\mathbb{1}_D - P_D)$ with $P_D = E_{N_D}(\{0\})$. Then we have

$$\begin{aligned}
(35) &\geq E_0(\tilde{H}_m) + M - M(\Psi, \tilde{Y}_{b,R}^* \tilde{X}_{f,R}^* (\mathbb{1}_D \otimes P_{\Omega_D} \otimes \mathbb{1}_{\text{rad}}) \tilde{X}_{f,R} \tilde{Y}_{b,R} \Psi) - \delta_f(R) \|\Psi\|_{2,1} \\
&\geq E_0(H_m) + M - M(\Psi, (\Gamma_f(q_{f,R}) \otimes \mathbb{1}_{\text{rad}}) \Psi) - \delta_{f,m}(R) \|\Psi\|_{2,1}.
\end{aligned}$$

Here we used $E_0(\tilde{H}_m) = E_0(H_m)$ and $X_{f,R}^*(\mathbb{1}_D \otimes P_D)X_{f,R} = \Gamma_f(q_{f,R})$ in the last line. Thus we have

$$\begin{aligned}
(\Psi, H_m \Psi) &\geq E_0(H_m) + m + M - M(\Psi, (\Gamma_f(q_{f,R}) \otimes \mathbb{1}_{\text{rad}}) \Psi) \\
&\quad - m(\Psi, (\mathbb{1}_D \otimes \Gamma_b(q_{b,R})) \Psi) - \delta_b(R) \|\Psi\|_{0,2} - \delta_f(R) \|\Psi\|_{2,1}. \tag{36}
\end{aligned}$$

Note that

$$\begin{aligned}
\mathbb{1}_D \otimes \mathbb{1}_{\text{rad}} &\geq \Gamma_f(q_{f,R}) \otimes \mathbb{1}_{\text{rad}} + (\mathbb{1}_D - \Gamma_f(q_{f,R})) \otimes \Gamma_b(q_{b,R}) \\
&= \Gamma_f(q_{f,R}) \otimes \mathbb{1}_{\text{rad}} + \mathbb{1}_D \otimes \Gamma_b(q_{b,R}) - \Gamma_f(q_{f,R}) \otimes \Gamma_b(q_{b,R}).
\end{aligned}$$

Then we have

$$\begin{aligned}
(\Psi, H_m \Psi) &\geq E_0(H_m) + m + (M - m)(\Psi, (\mathbb{1}_D \otimes \Gamma_b(q_{b,R})) \Psi) \\
&\quad - M(\Psi, (\Gamma_f(q_{f,R}) \otimes \Gamma_b(q_{b,R})) \Psi) - \delta_b(R) \|\Psi\|_{0,2} - \delta_f(R) \|\Psi\|_{2,1}.
\end{aligned}$$

Thus the proof is obtained. ■

Lemma 3.9 Assume (A.1) - (A.3). Then for all $0 < \varepsilon < \frac{1}{c_I |\kappa_I|}$,

$$\|H_{0,m} \Psi\| \leq L_\varepsilon \|H_m \Psi\| + R_\varepsilon \|\Psi\|, \quad \Psi \in \mathcal{D}(H_m),$$

where $L_\varepsilon = \frac{1}{1 - c_I |\kappa_I| \varepsilon}$ and $R_\varepsilon = \frac{1}{1 - c_I |\kappa_I| \varepsilon} (c_I |\kappa_I| (\frac{1}{2\varepsilon} + 1) + |\kappa_{II}| \|H_{II}\|)$.

(Proof) Let $\Psi \in \mathcal{D}(H_m)$. Since $H_{0,m} = H_m - \kappa_I H_I - \kappa_{II} H_{II}$, we see that

$$\|H_{0,m}\Psi\| \leq \|H_m\Psi\| + |\kappa_I| \|H_I\Psi\| + |\kappa_{II}| \|H_{II}\Psi\|.$$

From (30), it holds that $\|H_I\Psi\| \leq c_I\varepsilon\|H_{0,m}\Psi\| + c_I(\frac{1}{2\varepsilon} + 1)\|\Psi\|$ for all $\varepsilon > 0$. Hence

$$(1 - c_I|\kappa_I|\varepsilon)\|H_{0,m}\Psi\| \leq \|H_m\Psi\| + \left(c_I|\kappa_I|\left(\frac{1}{2\varepsilon} + 1\right) + |\kappa_{II}|\|H_{II}\|\right)\|\Psi\|. \quad (37)$$

Taking $\varepsilon > 0$ such that $\varepsilon < \frac{1}{c_I|\kappa_I|}$, we obtain the proof. ■

Since $\|N_D\Psi\| \leq \frac{1}{M}\|H_D\Psi\|$, $\Psi \in \mathcal{D}(H_D)$, and $\|N_{\text{rad}}\Phi\| \leq \frac{1}{m}\|H_{\text{rad}}\Phi\|$, $\Phi \in \mathcal{D}(H_{\text{rad}})$, the next corollary follows from Lemma 3.9.

Corollary 3.10 Assume **(A.1)** - **(A.3)**. Then for all $0 < \varepsilon < \frac{1}{c_I|\kappa_I|}$ and $\Psi \in \mathcal{D}(H_m)$,

$$\begin{aligned} \text{(i)} \quad \|(N_D \otimes \mathbb{1}_{\text{rad}})\Psi\| &\leq \frac{L_\varepsilon}{M}\|H_m\Psi\| + \frac{R_\varepsilon}{M}\|\Psi\|, \\ \text{(ii)} \quad \|(\mathbb{1}_D \otimes N_{\text{rad}})\Psi\| &\leq \frac{L_\varepsilon}{m}\|H_m\Psi\| + \frac{R_\varepsilon}{m}\|\Psi\|. \end{aligned}$$

(Proof of Theorem 3.4)

It is enough to show that $\sigma_{\text{ess}}(H_m) \subset [E_0(H_m) + m, \infty)$. Let $\lambda \in \sigma_{\text{ess}}(H_m)$. Then by the Weyl's theorem, there exists a sequence $\{\Psi_n\}_{n=1}^\infty$ of $\mathcal{D}(H_m)$ such that (i) $\|\Psi_n\| = 1$, $n \in \mathbf{N}$, (ii) $\text{s-lim}_{n \rightarrow \infty} (H_m - \lambda)\Psi_n = 0$, and (iii) $\text{w-lim}_{n \rightarrow \infty} \Psi_n = 0$. Since $|\lambda - (\Psi_n, H_m\Psi_n)| \leq |(\Psi_n, (H_m - \lambda)\Psi_n)| \leq \|(H_m - \lambda)\Psi_n\|$, it holds that $\lambda = \lim_{n \rightarrow \infty} (\Psi_n, H_m\Psi_n)$. Here we show that

$$\lim_{n \rightarrow \infty} (\Psi_n, H_m\Psi_n) \geq E_0(H_m) + m,$$

and then, the proof is obtained. Let $m \leq M$. From Lemma 3.8,

$$\begin{aligned} (\Psi_n, H_m\Psi_n) &\geq E_0(H_m) + m - M(\Psi_n, (\Gamma_f(q_{f,R}) \otimes \Gamma_b(q_{b,R})), \Psi_n) \\ &\quad - \delta_f(R)\|\Psi_n\|_{2,1} - \delta_{b,m}(R)\|\Psi_n\|_{0,2}. \end{aligned}$$

Since $\text{s-lim}_{n \rightarrow \infty} (H_m - \lambda)\Psi_n = 0$, we can set

$$E_m = \sup_{n \in \mathbf{N}} \|H_m\Psi_n\| < \infty.$$

Let $0 \leq \lambda \leq 2$ and $0 \leq \lambda' \leq 2$. From Corollary 3.10, it is seen that for all $0 < \varepsilon < \frac{1}{c_I|\kappa_I|}$,

$$\begin{aligned} \|\Psi_n\|_{\lambda, \lambda'} &= \|(N_D^{\lambda/2} \otimes \mathbb{1}_{\text{rad}})\Psi_n\| + \|(\mathbb{1}_D \otimes N_{\text{rad}}^{\lambda'})\Psi_n\| + \|\Psi_n\| \\ &\leq \|(N_D \otimes \mathbb{1}_{\text{rad}})\Psi_n\| + \|(\mathbb{1}_D \otimes N_{\text{rad}})\Psi_n\| + 3\|\Psi_n\| \\ &\leq \left(\frac{1}{M} + \frac{1}{m}\right)(L_\varepsilon\|H_m\Psi_n\| + 2R_\varepsilon) + 3\|\Psi_n\| \\ &\leq E_m L_\varepsilon \left(\frac{1}{M} + \frac{1}{m}\right) + 2\left(\frac{1}{M} + \frac{1}{m}\right)R_\varepsilon + 3. \end{aligned}$$

Then we have

$$(\Psi_n, H_m \Psi_n) \geq E_0(H_m) + m - M(\Psi_n, (\Gamma_f(q_{f,R}) \otimes \Gamma_b(q_{b,R})), \Psi_n) - \delta_{m,\varepsilon}(R), \quad (38)$$

where $\delta_{m,\varepsilon}(R) = c_{m,\varepsilon}(\delta_b(R) + \delta_f(R))$ with $c_{m,\varepsilon} = E_m L_\varepsilon (\frac{1}{M} + \frac{1}{m}) + 2(\frac{1}{M} + \frac{1}{m})R_\varepsilon + 3$. It is seen that

$$\begin{aligned} & |(\Psi_n, (\Gamma_f(q_{f,R}) \otimes \Gamma_b(q_{b,R}))) \Psi_n| \\ & \leq \|(H_{0,m} + 1)^{1/2} \Psi_n\| \|(H_{0,m} + 1)^{-1/2} (\Gamma_f(q_{f,R}) \otimes \Gamma_b(q_{b,R})) \Psi_n\|. \end{aligned} \quad (39)$$

From Lemma 3.9, we see that

$$\|(H_{0,m} + 1)^{1/2} \Psi_n\| \leq \|H_{0,m} \Psi_n\| + \|\Psi_n\| \leq L_\varepsilon \|H_m \Psi_n\| + (R_\varepsilon + 1) \|\Psi_n\| = E_0(H_m) L_\varepsilon + R_\varepsilon + 1,$$

and hence,

$$\sup_{n \in \mathbb{N}} \|(H_{0,m} + 1)^{1/2} \Psi_n\| \leq E_m L_\varepsilon + R_\varepsilon + 1. \quad (40)$$

It holds that

$$\begin{aligned} (H_{0,m} + 1)^{-1/2} (\Gamma_f(q_{f,R}) \otimes \Gamma_b(q_{b,R})) &= (H_{0,m} + 1)^{-1/2} ((H_D + 1)^{1/2} \otimes (H_{\text{rad},m} + 1)^{1/2}) \\ &\quad \times ((H_D + 1)^{-1/2} \Gamma_f(q_{f,R})) \otimes ((H_{\text{rad},m} + 1)^{-1/2} \Gamma_b(q_{b,R})), \end{aligned}$$

and hence, $(H_{0,m} + 1)^{-1/2} (\Gamma_f(q_{f,R}) \otimes \Gamma_b(q_{b,R}))$ is compact, since $\|(H_{0,m} + 1)^{-1/2} ((H_D + 1)^{1/2} \otimes (H_{\text{rad},m} + 1)^{1/2})\| \leq 1$ and $((H_D + 1)^{-1/2} \Gamma_f(q_{f,R})) \otimes ((H_{\text{rad},m} + 1)^{-1/2} \Gamma_b(q_{b,R}))$ is compact. Therefore it holds that

$$\lim_{n \rightarrow \infty} \|(H_{0,m} + 1)^{-1/2} (\Gamma_f(q_{f,R}) \otimes \Gamma_b(q_{b,R})) \Psi_n\| = 0. \quad (41)$$

From (39) - (41) we have $\lim_{n \rightarrow \infty} |(\Psi_n, (\Gamma_f(q_{f,R}) \otimes \Gamma_b(q_{b,R}))) \Psi_n| = 0$. Then by taking the limit of (38) as $R \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} (\Psi_n, H_m \Psi_n) \geq E_0(H_m) + m$. ■

4 Derivative Bounds

From Theorem 3.4, H_m has the ground state. Let Ψ_m be the normalized ground state of H_m , i.e.

$$H_m \Psi_m = E_0(H_m) \Psi_m, \quad \|\Psi_m\| = 1.$$

4.1 Electron-Positron Derivative Bounds

We introduce the distribution kernel of the annihilation operator for the Dirac field. For all $\Psi = \{\Psi^{(n)} = {}^t(\Psi_1^{(n)}, \dots, \Psi_4^{(n)})\}_{n=0}^\infty \in \mathcal{D}(H_D)$, we set

$$C_l(\mathbf{p}) \Psi^{(n,v)}(\mathbf{p}_1, \dots, \mathbf{p}_n) = \delta_{l,v} \sqrt{n+1} \Psi^{(n+1,v)}(\mathbf{p}, \mathbf{p}_1, \dots, \mathbf{p}_n). \quad l = 1, \dots, 4.$$

Let

$$b_{1/2}(\mathbf{p}) = C_1(\mathbf{p}), \quad b_{-1/2}(\mathbf{p}) = C_2(\mathbf{p}), \quad d_{1/2}(\mathbf{p}) = C_3(\mathbf{p}), \quad d_{-1/2}(\mathbf{p}) = C_4(\mathbf{p}).$$

Then it follows that for all $\Phi \in \mathcal{F}_{\text{Dirac}}$ and $\Psi \in \mathcal{D}(H_D)$,

$$\begin{aligned}(\Phi, b_s(f)\Psi) &= \int_{\mathbf{R}^3} f(\mathbf{p})^* (\Phi, b_s(\mathbf{p})\Psi) d\mathbf{p}, & f \in \mathcal{D}(\omega_M), \\(\Phi, d_s(g)\Psi) &= \int_{\mathbf{R}^3} g(\mathbf{p})^* (\Phi, d_s(\mathbf{p})\Psi) d\mathbf{p}, & g \in \mathcal{D}(\omega_M).\end{aligned}$$

The number operator for electrons and positrons are defined by

$$N_D^+ = d\Gamma_f \left(\begin{pmatrix} \mathbb{1} & O \\ O & O \end{pmatrix} \right), \quad N_D^- = d\Gamma_f \left(\begin{pmatrix} O & O \\ O & \mathbb{1} \end{pmatrix} \right),$$

respectively. It holds that for all $\Phi, \Psi \in \mathcal{D}(H_D)$,

$$\begin{aligned}(\Phi, N_D^+ \Psi) &= \sum_{\pm 1/2} \int_{\mathbf{R}^3} (b_s(\mathbf{p})\Phi, b_s(\mathbf{p})\Psi) d\mathbf{p}, \\(\Phi, N_D^- \Psi) &= \sum_{\pm 1/2} \int_{\mathbf{R}^3} (d_s(\mathbf{p})\Phi, d_s(\mathbf{p})\Psi) d\mathbf{p}.\end{aligned}$$

By the canonical anti-commutation relation, it is proven in ([22] ; Section III) that

$$[\psi^\dagger(\mathbf{x}) \alpha^j \psi(\mathbf{x}), b_s(f)] = - \sum_{l, l'=1}^4 \alpha_{l, l'}^j(f, f_{s, \mathbf{x}}^l) \psi_{l'}(\mathbf{x}), \quad (42)$$

$$[\psi^\dagger(\mathbf{x}) \alpha^j \psi(\mathbf{x}), d_s(g)] = \sum_{l, l'=1}^4 \alpha_{l, l'}^j(g, g_{s, \mathbf{x}}^{l'}) \psi_l(\mathbf{x})^*, \quad (43)$$

and for $\rho(\mathbf{x}) = \psi^\dagger(\mathbf{x})\psi(\mathbf{x})$,

$$[\rho(\mathbf{x})\rho(\mathbf{y}), b_s(f)] = - \sum_{l=1}^4 \left((f, f_{s, \mathbf{y}}^l) \rho(\mathbf{x}) \psi_l(\mathbf{y}) + (f, f_{s, \mathbf{x}}^l) \psi_l(\mathbf{x}) \rho(\mathbf{y}) \right), \quad (44)$$

$$[\rho(\mathbf{x})\rho(\mathbf{y}), d_s(g)] = \sum_{l=1}^4 \left((g, g_{s, \mathbf{y}}^l) \rho(\mathbf{x}) \psi_l(\mathbf{y})^* + (g, g_{s, \mathbf{x}}^l) \psi_l(\mathbf{x})^* \rho(\mathbf{y}) \right). \quad (45)$$

Let X and Y be operators on a Hilbert space. The weak commutator is defined by

$$[X, Y]^0(\Phi, \Psi) = (X^* \Phi, Y \Psi) - (Y^* \Phi, X \Psi),$$

where $\Psi \in \mathcal{D}(X) \cap \mathcal{D}(Y)$ and $\Phi \in \mathcal{D}(X^*) \cap \mathcal{D}(Y^*)$.

Lemma 4.1 Assume (A.1) - (A.3). Then it holds that for all $f \in L^2(\mathbf{R}^3)$,

$$(i) [H_I, b_s(f) \otimes \mathbb{1}]^0(\Phi, \Psi) = \int_{\mathbf{R}^3} f(\mathbf{p})^* (\Phi, K_s^+(\mathbf{p})\Psi) d\mathbf{p}, \quad \Phi \in \mathcal{F}_{\text{QED}}, \Psi \in \mathcal{D}(H_m),$$

$$(ii) [H_{II}, b_s(f) \otimes \mathbb{1}]^0(\Phi, \Psi) = \int_{\mathbf{R}^3} f(\mathbf{p})^* (\Phi, (S_s^+(\mathbf{p}) + T_s^+(\mathbf{p}))\Psi) d\mathbf{p}, \quad \Phi, \Psi \in \mathcal{F}_{\text{QED}}.$$

Here $K_s^+(\mathbf{p})$, $S_s^+(\mathbf{p})$ and $T_s^+(\mathbf{p})$ are operators which satisfy

$$\begin{aligned}(\Phi, K_s^+(\mathbf{p})\Psi) &= - \sum_{j=1}^3 \sum_{l,l'=1}^4 \alpha_{l,l'}^j \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) f_{s,\mathbf{x}}^l(\mathbf{p}) (\Phi, (\psi_{l'}(\mathbf{x}) \otimes A_j(\mathbf{x}))\Psi) d\mathbf{x}, \\(\Phi, S_s^+(\mathbf{p})\Psi) &= - \sum_{l=1}^4 \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{II}(\mathbf{x}) \chi_{II}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} f_{s,\mathbf{y}}^l(\mathbf{p}) (\Phi, (\rho(\mathbf{x}) \psi_l(\mathbf{y}) \otimes \mathbb{1}))\Psi) d\mathbf{x} d\mathbf{y}, \\(\Phi, T_s^+(\mathbf{p})\Psi) &= - \sum_{l=1}^4 \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{II}(\mathbf{x}) \chi_{II}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} f_{s,\mathbf{x}}^l(\mathbf{p}) (\Phi, (\psi_l(\mathbf{x}) \rho(\mathbf{y}) \otimes \mathbb{1}))\Psi) d\mathbf{x} d\mathbf{y}.\end{aligned}$$

(Proof)

(i) Let $\Phi \in \mathcal{F}_{\text{QED}}$ and $\Psi \in \mathcal{D}(H_m)$. By (42), we have

$$\begin{aligned}[H_I, b_s(f) \otimes \mathbb{1}]^0(\Phi, \Psi) &= \sum_{j=1}^3 \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) [\psi^\dagger(\mathbf{x}) \alpha^j \psi(\mathbf{x}) \otimes A_j(\mathbf{x}), b_s(f) \otimes \mathbb{1}]^0(\Phi, \Psi) d\mathbf{x} \\&= \sum_{j=1}^3 \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) (\Phi, ([\psi^\dagger(\mathbf{x}) \alpha^j \psi(\mathbf{x}), b_s(f)] \otimes A_j(\mathbf{x})) \Psi) d\mathbf{x} \\&= - \sum_{j=1}^3 \sum_{l,l'=1}^4 \alpha_{l,l'}^j \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) (f, f_{s,\mathbf{x}}^l) (\Phi, (\psi_{l'}(\mathbf{x}) \otimes A_j(\mathbf{x}))\Psi) d\mathbf{x}.\end{aligned}$$

Let $\ell_{s,\mathbf{p}} : \mathcal{F}_{\text{QED}} \times \mathcal{F}_{\text{QED}} \rightarrow \mathbf{C}$ be a functional defined by

$$\ell_{s,\mathbf{p}}(\Phi', \Psi') = - \sum_{j=1}^3 \sum_{l,l'=1}^4 \alpha_{l,l'}^j \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) f_{s,\mathbf{x}}^l(\mathbf{p}) (\Phi', (\psi_{l'}(\mathbf{x}) \otimes A_j(\mathbf{x}))\Psi') d\mathbf{x},$$

for $\Phi' \in \mathcal{F}_{\text{QED}}$, $\Psi' \in \mathcal{D}(H_{0,m})$. We see that

$$\ell_{s,\mathbf{p}}(\Phi', \Psi') \leq c_{I,s,\mathbf{p}} \|\Phi'\| \|(\mathbb{1} \otimes (H_{\text{rad},m} + 1)^{1/2})\Psi'\|,$$

where $c_{I,s,\mathbf{p}} = \sum_{j=1}^3 \sum_{l,l'=1}^4 |\alpha_{l,l'}^j| \|\chi_I\|_{L^1} |f_s^l(\mathbf{p})| c_D^{l'} c_{\text{rad}}^j$. Then from the Riesz Representation theorem, we can define an operator $K_s^+(\mathbf{p})$ which satisfy $\ell_{s,\mathbf{p}}(\Phi', \Psi') = (\Phi', K_s^+(\mathbf{p})\Psi')$. Then it holds that

$$[H_I, b_s(f)]^0(\Phi, \Psi) = \int_{\mathbf{R}^3} f(\mathbf{p})^* \ell_{s,\mathbf{p}}(\Phi, \Psi) d\mathbf{p} = \int_{\mathbf{R}^3} f(\mathbf{p})^* (\Phi, K_s^+(\mathbf{p})\Psi) d\mathbf{p}.$$

(ii) From (44), we see that for all $\Phi, \Psi \in \mathcal{F}_{\text{QED}}$,

$$\begin{aligned}[H_{II}, b_s(f) \otimes \mathbb{1}]^0(\Phi, \Psi) &= \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{II}(\mathbf{x}) \chi_{II}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} [\rho(\mathbf{x}) \rho(\mathbf{y}) \otimes \mathbb{1}, b_s(f) \otimes \mathbb{1}]^0(\Phi, \Psi) d\mathbf{x} d\mathbf{y} \\&= \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{II}(\mathbf{x}) \chi_{II}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} (\Phi, ([\rho(\mathbf{x}) \rho(\mathbf{y}), b_s(f)] \otimes \mathbb{1}) \Psi) d\mathbf{x} d\mathbf{y} \\&= - \sum_{l=1}^4 \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{II}(\mathbf{x}) \chi_{II}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \left\{ (f, f_{s,\mathbf{y}}^l) (\Phi, (\rho(\mathbf{x}) \psi_l(\mathbf{y}) \otimes \mathbb{1}) \Psi) \right. \\&\quad \left. + (f, f_{s,\mathbf{x}}^l) (\Phi, (\psi_l(\mathbf{x}) \rho(\mathbf{y}) \otimes \mathbb{1}) \Psi) \right\} d\mathbf{x} d\mathbf{y}.\end{aligned}$$

We set functionals $q_{s,\mathbf{p}}$ and $r_{s,\mathbf{p}}$ on $\mathcal{F}_{\text{QED}} \times \mathcal{F}_{\text{QED}}$ by

$$\begin{aligned} q_{s,\mathbf{p}}(\Phi', \Psi') &= - \sum_{l=1}^4 \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{\Pi}(\mathbf{x})\chi_{\Pi}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} f_{s,\mathbf{y}}^l(\mathbf{p})(\Phi', (\rho(\mathbf{x})\psi_l(\mathbf{y}) \otimes \mathbb{1})\Psi') d\mathbf{x}d\mathbf{y}, \quad \Phi', \Psi' \in \mathcal{F}_{\text{QED}}, \\ r_{s,\mathbf{p}}(\Phi'', \Psi'') &= - \sum_{l=1}^4 \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{\Pi}(\mathbf{x})\chi_{\Pi}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} f_{s,\mathbf{x}}^l(\mathbf{p})(\Phi'', (\psi_l(\mathbf{x})\rho(\mathbf{y}) \otimes \mathbb{1})\Psi'') d\mathbf{x}d\mathbf{y}, \quad \Phi'', \Psi'' \in \mathcal{F}_{\text{QED}}. \end{aligned}$$

We see that

$$\begin{aligned} q_{s,\mathbf{p}}(\Phi', \Psi') &\leq c_{\Pi,s,\mathbf{p}} \|\Phi'\| \|\Psi'\|, \\ r_{s,\mathbf{p}}(\Phi'', \Psi'') &\leq c_{\Pi,s,\mathbf{p}} \|\Phi''\| \|\Psi''\|, \end{aligned}$$

where $c_{\Pi,s,\mathbf{p}} = \sum_{l,l'=1}^4 \left\| \frac{\chi_{\Pi}(\mathbf{x})\chi_{\Pi}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \right\|_{L^1} |f_s^l(\mathbf{p})| (c_D'')^2 c_D^l$. Then from Riesz Representation theorem, we can define operators $S_s^+(\mathbf{p})$ and $T_s^+(\mathbf{p})$ such that $q_{s,\mathbf{p}}(\Phi', \Psi') = (\Phi', S_s^+(\mathbf{p})\Psi')$ and $r_{\mathbf{p}}(\Phi'', \Psi'') = (\Phi'', T_s^+(\mathbf{p})\Psi'')$, respectively. Then it holds that

$$\begin{aligned} [H_{\Pi}, b_s(f)]^0(\Phi, \Psi) &= \int_{\mathbf{R}^3} \overline{f(\mathbf{p})} (q_{s,\mathbf{p}}(\Phi, \Psi) + r_{s,\mathbf{p}}(\Phi, \Psi)) d\mathbf{p} \\ &= \int_{\mathbf{R}^3} \overline{f(\mathbf{p})} (\Phi, (S_s^+(\mathbf{p}) + T_s^+(\mathbf{p}))\Psi) d\mathbf{p}. \end{aligned}$$

Thus proof is obtained. ■

In a similar way to Lemma 4.1, the following lemma is also proven.

Lemma 4.2 Assume (A.1) - (A.3). Then it holds that for all $g \in L^2(\mathbf{R}^3)$,

$$\begin{aligned} \text{(i)} \quad [H_I, d_s(g) \otimes \mathbb{1}]^0(\Phi, \Psi) &= \int_{\mathbf{R}^3} g(\mathbf{p})^* (\Phi, K_s^-(\mathbf{p})\Psi) d\mathbf{p}, \quad \Phi \in \mathcal{F}_{\text{QED}}, \Psi \in \mathcal{D}(H_m), \\ \text{(ii)} \quad [H_{\Pi}, d_s(g) \otimes \mathbb{1}]^0(\Phi, \Psi) &= \int_{\mathbf{R}^3} g(\mathbf{p})^* (\Phi, (S_s^-(\mathbf{p}) + T_s^-(\mathbf{p}))\Psi) d\mathbf{p}, \quad \Phi, \Psi \in \mathcal{F}_{\text{QED}}. \end{aligned}$$

Here $K_s^-(\mathbf{p})$, $S_s^-(\mathbf{p})$ and $T_s^-(\mathbf{p})$ are operators which satisfy

$$\begin{aligned} (\Phi, K_s^-(\mathbf{p})\Psi) &= \sum_{j=1}^3 \sum_{l,l'=1}^4 \alpha_{l,l'}^j \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) g_{s,\mathbf{x}}^{l'}(\mathbf{p}) (\Phi, (\psi_l(\mathbf{x})^* \otimes A_j(\mathbf{x}))\Psi) d\mathbf{x}, \\ (\Phi, S_s^-(\mathbf{p})\Psi) &= \sum_{l=1}^4 \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{\Pi}(\mathbf{x})\chi_{\Pi}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} g_{s,\mathbf{y}}^l(\mathbf{p}) (\Phi, (\rho(\mathbf{x})\psi_l(\mathbf{y})^* \otimes \mathbb{1}))\Psi) d\mathbf{x}d\mathbf{y}, \\ (\Phi, T_s^-(\mathbf{p})\Psi) &= \sum_{l=1}^4 \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{\Pi}(\mathbf{x})\chi_{\Pi}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} g_{s,\mathbf{x}}^l(\mathbf{p}) (\Phi, (\psi_l(\mathbf{x})^* \rho(\mathbf{y}) \otimes \mathbb{1}))\Psi) d\mathbf{x}d\mathbf{y}, \end{aligned}$$

Lemma 4.3 Assume (A.1) - (A.5). Let $\Psi \in \mathcal{D}(H_m)$. Then, $K_s^\pm(\mathbf{p})\Psi$, $S_s^\pm(\mathbf{p})\Psi$ and $T_s^\pm(\mathbf{p})\Psi$, $s = \pm 1/2$, are strongly differentiable for all $\mathbf{p} \in \mathbf{R}^3 \setminus O_D$.

(Proof) We show that $K_s^+(\mathbf{p})\Psi$ is strongly differentiable. Let $\Phi \in \mathcal{F}_{\text{QED}}$ with $\|\Phi\| = 1$. From (A.4), $K_s^+(\mathbf{p})\Psi$ is weakly differentiable for all $\mathbf{p} \in \mathbf{R}^3 \setminus O_D$, and we have

$$\partial_{p^\nu}(\Phi, K_s^+(\mathbf{p})\Psi) = - \sum_{j=1}^3 \sum_{l,l'=1}^4 \alpha_{l,l'}^j \int_{\mathbf{R}^3} \chi_l(\mathbf{x}) \partial_{p^\nu} f_{s,\mathbf{x}}^l(\mathbf{p}) (\Phi, (\psi_{l'}(\mathbf{x}) \otimes A_j(\mathbf{x}))\Psi) d\mathbf{x},$$

and $|\partial_{p^\nu}(\Phi, K_s^\pm(\mathbf{p})\Psi)| \leq \sum_{j=1}^3 \sum_{l,l'=1}^4 |\alpha_{l,l'}^j| c_D^{l'} c_{\text{rad}}^j (\int_{\mathbf{R}^3} |\partial_{p^\nu} f_{s,\mathbf{x}}^l(\mathbf{p})| d\mathbf{x}) \|(\mathbb{1} \otimes H_{\text{rad},m}^{1/2})\Psi\|$. Then the Riesz representation theorem shows that there exists a vector $\Xi_\Psi(\mathbf{p}) \in \mathcal{F}_{\text{QED}}$ such that $(\Phi, \Xi_\Psi(\mathbf{p})) = \partial_{p^\nu}(\Phi, K_s^\pm(\mathbf{p})\Psi)$. Let $\mathbf{e}_\nu = (\delta_{\nu,j})_{j=1}^3$. It is seen that

$$\begin{aligned} & (\Phi, \frac{K_s^+(\mathbf{p} + \varepsilon \mathbf{e}_\nu) - K_s^+(\mathbf{p})}{\varepsilon} \Psi) - (\Phi, \Xi(\mathbf{p})) \\ &= - \sum_{j=1}^3 \sum_{l,l'=1}^4 \alpha_{l,l'}^j \int_{\mathbf{R}^3} \chi_l(\mathbf{x}) \left(\frac{f_{s,\mathbf{x}}^l(\mathbf{p} + \varepsilon \mathbf{e}_\nu) - f_{s,\mathbf{x}}^l(\mathbf{p})}{\varepsilon} - \partial_{p^\nu} f_{s,\mathbf{x}}^l(\mathbf{p}) \right) (\Phi, (\psi_{l'}(\mathbf{x}) \otimes A_j(\mathbf{x}))\Psi) d\mathbf{x}, \end{aligned}$$

and hence,

$$\begin{aligned} & |(\Phi, (\frac{K_s^+(\mathbf{p} + \varepsilon \mathbf{e}_\nu) - K_s^+(\mathbf{p})}{\varepsilon} \Psi - \Xi(\mathbf{p}))| \\ & \leq \sum_{j=1}^3 \sum_{l,l'=1}^4 |\alpha_{l,l'}^j| c_D^{l'} c_{\text{rad}}^{j'} \left(\int_{\mathbf{R}^3} |\chi_l(\mathbf{x})| \left| \frac{f_{s,\mathbf{x}}^l(\mathbf{p} + \varepsilon \mathbf{e}_\nu) - f_{s,\mathbf{x}}^l(\mathbf{p})}{\varepsilon} - \partial_{p^\nu} f_{s,\mathbf{x}}^l(\mathbf{p}) \right| d\mathbf{x} \right) \|\Psi\|. \end{aligned} \quad (46)$$

Since (46) holds for all $\Phi \in \mathcal{F}_{\text{QED}}$ with $\|\Phi\| = 1$, we have

$$\begin{aligned} & \left\| \frac{K_s^+(\mathbf{p} + \varepsilon \mathbf{e}_\nu) - K_s^+(\mathbf{p})}{\varepsilon} \Psi - \Xi(\mathbf{p}) \right\| \\ & \leq \sum_{j=1}^3 \sum_{l,l'=1}^4 |\alpha_{l,l'}^j| c_D^{l'} c_{\text{rad}}^{j'} \left(\int_{\mathbf{R}^3} |\chi_l(\mathbf{x})| \left| \frac{f_{s,\mathbf{x}}^l(\mathbf{p} + \varepsilon \mathbf{e}_\nu) - f_{s,\mathbf{x}}^l(\mathbf{p})}{\varepsilon} - \partial_{p^\nu} f_{s,\mathbf{x}}^l(\mathbf{p}) \right| d\mathbf{x} \right) \|\Psi\| \rightarrow 0, \end{aligned}$$

as $\varepsilon \rightarrow 0$. Thus $K_s^+(\mathbf{p})\Psi$ is strongly differentiable. Similarly, it is proven that $K_s^-(\mathbf{p})\Psi$, $S_s^\pm(\mathbf{p})\Psi$ and $T_s^\pm(\mathbf{p})\Psi$ are strongly differentiable for all $\mathbf{p} \in \mathbf{R}^3 \setminus O_D$. ■

Lemma 4.4 For all $\Phi, \Psi \in \mathcal{D}(H_D)$, it holds that

- (i) $[H_D, b_s(f)]^0(\Phi, \Psi) = -(\Phi, b_s(\omega_M f)\Psi),$
- (ii) $[H_D, d_s(f)]^0(\Phi, \Psi) = -(\Phi, d_s(\omega_M f)\Psi).$

(Proof) It holds that for all $\Phi \in \mathcal{F}_{\text{Dirac}}^{\text{fin}}(\mathcal{D}(\omega_M))$,

$$[H_D, b_s^\dagger(f)]\Phi = b_s^\dagger(\omega_M f)\Phi.$$

Let $\Psi \in \mathcal{D}(H_m)$. Then

$$(H_D \Phi, b_s(f)\Psi) - (b_s(f)\Phi, H_D \Psi) = ([b_s^\dagger(f), H_D]\Phi, \Psi) = (-b_s^\dagger(\omega_M f)\Phi, \Psi),$$

and hence,

$$(H_D \Phi, b_s(f) \Psi) - (b_s(f) \Phi, H_D \Psi) = -(\Phi, b_s(\omega_M f) \Psi). \quad (47)$$

Since $\mathcal{F}_{\text{Dirac}}^{\text{fin}}(\mathcal{D}(\omega_M))$ is a core of H_D and $b_s(f)$ is bounded, (47) holds for all $\Phi \in \mathcal{D}(H_D)$. Hence (i) follows. Similarly, we can also prove (ii). ■

Proposition 4.5 (Electron-Positron Pull-Through Formula)

Assume (A.1) - (A.3). Then that

$$\begin{aligned} \text{(i)} \quad & (b_s(\mathbf{p}) \otimes \mathbb{1}) \Psi_m = (H_m - E_0(H_m) + \omega_M(\mathbf{p}))^{-1} (\kappa_I K_s^+(\mathbf{p}) + \kappa_{II} (S_s^+(\mathbf{p}) + T_s^+(\mathbf{p})) \Psi_m, \\ \text{(ii)} \quad & (d_s(\mathbf{p}) \otimes \mathbb{1}) \Psi_m = (H_m - E_0(H_m) + \omega_M(\mathbf{p}))^{-1} (\kappa_I K_s^-(\mathbf{p}) + \kappa_{II} (S_s^-(\mathbf{p}) + T_s^-(\mathbf{p})) \Psi_m, \end{aligned}$$

for almost everywhere $\mathbf{p} \in \mathbf{R}^3$.

(Proof) Let $\Phi \in \mathcal{D}(H_m)$. By Lemma 4.4 (i), we have

$$\begin{aligned} & [H_m, b_s(f) \otimes \mathbb{1}]^0(\Phi, \Psi_m) \\ &= -(\Phi, (b_s(\omega_M f) \otimes \mathbb{1}) \Psi_m) + \kappa_I [H_I, b_s(f) \otimes \mathbb{1}]^0(\Phi, \Psi_m) + \kappa_{II} [H_{II}, b_s(f) \otimes \mathbb{1}]^0(\Phi, \Psi_m). \end{aligned}$$

On the other hand, $H_m \Psi_m = E_0(H_m) \Psi_m$ yields that

$$[H_m, b_s(f) \otimes \mathbb{1}]^0(\Phi, \Psi_m) = ((H_m - E_0(H_m)) \Phi, (b_s(f) \otimes \mathbb{1}) \Psi_m).$$

Then, we have

$$\begin{aligned} & ((H_m - E_0(H_m)) \Phi, (b_s(f) \otimes \mathbb{1}) \Psi_m) + (\Phi, (b_s(\omega_M f) \otimes \mathbb{1}) \Psi_m) \\ &= \kappa_I [H_I, b_s(f) \otimes \mathbb{1}]^0(\Phi, \Psi_m) + \kappa_{II} [H_{II}, b_s(f) \otimes \mathbb{1}]^0(\Phi, \Psi_m). \end{aligned}$$

By Lemma 4.1, it follows that

$$\begin{aligned} & \int_{\mathbf{R}^3} f(\mathbf{p})^* \left((H_m - E_0(H_m) + \omega_M(\mathbf{p})) \Phi, (b_s(\mathbf{p}) \otimes \mathbb{1}) \Psi_m \right) d\mathbf{p} \\ &= \int_{\mathbf{R}^3} f(\mathbf{p})^* (\Phi, (\kappa_I K_s^+(\mathbf{p}) + \kappa_{II} (S_s^+(\mathbf{p}) + T_s^+(\mathbf{p})) \Psi_m) d\mathbf{p}. \end{aligned} \quad (48)$$

Since (48) holds for all $f \in L^2(\mathbf{R}^3)$, it follows that

$$((H_m - E_0(H_m) + \omega_M(\mathbf{p})) \Phi, (b_s(\mathbf{p}) \otimes \mathbb{1}) \Psi_m) = (\Phi, (\kappa_I K_s^+(\mathbf{p}) + \kappa_{II} (S_s^+(\mathbf{p}) + T_s^+(\mathbf{p})) \Psi_m),$$

for almost everywhere $\mathbf{p} \in \mathbf{R}^3$. This implies that $(b_s(\mathbf{p}) \otimes \mathbb{1}) \Psi_m \in \mathcal{D}(H_m)$ and

$$(H_m - E_0(H_m) + \omega_M(\mathbf{p})) (b_s(\mathbf{p}) \otimes \mathbb{1}) \Psi_m = (\kappa_I K_s^+(\mathbf{p}) + \kappa_{II} (S_s^+(\mathbf{p}) + T_s^+(\mathbf{p})) \Psi_m. \quad (49)$$

From (49), we obtain (i). Similarly, (ii) is also proven. ■

Theorem 4.6 (Electron-Positron Derivative Bounds)

Assume (A.1) - (A.5). Then, it holds that for all $\mathbf{p} \in \mathbf{R}^3 \setminus O_D$ and $0 < \varepsilon < \frac{1}{c_1 |\kappa_1|}$,

$$\begin{aligned} \text{(i)} \quad & \|\partial_{p^\nu}(b_s(\mathbf{p}) \otimes \mathbf{1})\Psi_m\| \leq \left((L_\varepsilon E_0(H_m) + R_\varepsilon + 1)|\kappa_1| + 2|\kappa_{II}| \right) F_{s,+}^\nu(\mathbf{p}), \\ \text{(ii)} \quad & \|\partial_{p^\nu}(d_s(\mathbf{p}) \otimes \mathbf{1})\Psi_m\| \leq \left((L_\varepsilon E_0(H_m) + R_\varepsilon + 1)|\kappa_1| + 2|\kappa_{II}| \right) F_{s,-}^\nu(\mathbf{p}). \end{aligned}$$

Here $F_{s,\pm}^\nu$ are functions satisfying $F_{s,\pm}^\nu \in L^2(\mathbf{R}^3)$, $s = \pm 1/2$, $\nu = 1, \dots, 3$.

(Proof) Let $R_{m,M}(\mathbf{p}) = (H_m - E_0(H_m) + \omega_M(\mathbf{p}))^{-1}$. From Proposition 4.5 it holds that for all $\Phi \in \mathcal{F}_{\text{QED}}$ with $\|\Phi\| = 1$,

$$\begin{aligned} (\Phi, \partial_{p^\nu}(b_s(\mathbf{p}) \otimes \mathbf{1})\Psi_m) &= \kappa_I (\Phi, \partial_{p^\nu} R_{m,M}(\mathbf{p}) K_s^+(\mathbf{p}) \Psi_m) + \kappa_{II} (\Phi, \partial_{p^\nu} R_{m,M}(\mathbf{p}) S_s^+(\mathbf{p}) \Psi_m) \\ &\quad + \kappa_{II} (\Phi, \partial_{p^\nu} R_{m,M}(\mathbf{p}) T_s^+(\mathbf{p}) \Psi_m). \end{aligned} \quad (50)$$

Here we evaluate the three terms in the right-hand side of (50) as follows.

(First term) We see that

$$\begin{aligned} & (\Phi, \partial_{p^\nu} R_{m,M}(\mathbf{p}) K_s^+(\mathbf{p}) \Psi_m) \\ &= - \sum_{j=1}^3 \sum_{l,l'=1}^4 \alpha_{l,l'}^j \partial_{p^\nu} \left(f_s^l(\mathbf{p}) \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) e^{-i\mathbf{p} \cdot \mathbf{x}} (R_{m,M}(\mathbf{p}) \Phi, (\psi_{l'}(\mathbf{x}) \otimes A_j(\mathbf{x})) \Psi_m) d\mathbf{x} \right), \\ &= - \sum_{j=1}^3 \sum_{l,l'=1}^4 \alpha_{l,l'}^j \left\{ (\partial_{p^\nu} f_s^l(\mathbf{p})) \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) e^{-i\mathbf{p} \cdot \mathbf{x}} (R_{m,M}(\mathbf{p}) \Phi, (\psi_{l'}(\mathbf{x}) \otimes A_j(\mathbf{x})) \Psi_m) d\mathbf{x} \right. \\ &\quad \left. - i f_s^l(\mathbf{p}) \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) x^\nu e^{-i\mathbf{p} \cdot \mathbf{x}} (R_{m,M}(\mathbf{p}) \Phi, (\psi_{l'}(\mathbf{x}) \otimes A_j(\mathbf{x})) \Psi_m) d\mathbf{x} \right. \\ &\quad \left. - \frac{f_s^l(\mathbf{p}) p^\nu}{\omega_M(\mathbf{p})} \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) e^{-i\mathbf{p} \cdot \mathbf{x}} (R_{m,M}(\mathbf{p})^2 \Phi, (\psi_{l'}(\mathbf{x}) \otimes A_j(\mathbf{x})) \Psi_m) d\mathbf{x} \right\}. \end{aligned}$$

Since $\|R_{m,M}(\mathbf{p})\| \leq \frac{1}{\omega_M(\mathbf{p})} \leq \frac{1}{M}$ and $\|\Phi\| = 1$, we have

$$\begin{aligned} \left| \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) e^{-i\mathbf{p} \cdot \mathbf{x}} (R_{m,M}(\mathbf{p}) \Phi, (\psi_{l'}(\mathbf{x}) \otimes A_j(\mathbf{x})) \Psi_m) d\mathbf{x} \right| &\leq \frac{c_D^{l'} c_{\text{rad}}^j}{M} \|\chi_I\|_{L^1} \|(\mathbf{1} \otimes (H_{\text{rad},m} + 1)^{1/2}) \Psi_m\|, \\ \left| \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) x^\nu e^{-i\mathbf{p} \cdot \mathbf{x}} (R_{m,M}(\mathbf{p}) \Phi, (\psi_{l'}(\mathbf{x}) \otimes A_j(\mathbf{x})) \Psi_m) d\mathbf{x} \right| &\leq \frac{c_D^{l'} c_{\text{rad}}^j}{M} \|\mathbf{x} \chi_I\|_{L^1} \|(\mathbf{1} \otimes (H_{\text{rad},m} + 1)^{1/2}) \Psi_m\|, \\ \left| \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) e^{-i\mathbf{p} \cdot \mathbf{x}} (R_{m,M}(\mathbf{p})^2 \Phi, (\psi_{l'}(\mathbf{x}) \otimes A_j(\mathbf{x})) \Psi_m) d\mathbf{x} \right| &\leq \frac{c_D^{l'} c_{\text{rad}}^j}{M^2} \|\chi_I\|_{L^1} \|(\mathbf{1} \otimes (H_{\text{rad},m} + 1)^{1/2}) \Psi_m\|. \end{aligned}$$

It is seen that $\|(\mathbf{1} \otimes (H_{\text{rad},m}^{1/2} + 1)^{1/2}) \Psi_m\| \leq \|H_{0,m} \Psi_m\| + \|\Psi_m\| = \|H_{0,m} \Psi_m\| + 1$, and hence,

$$\begin{aligned} & |\partial_{p^\nu} (\Phi, R_{m,M}(\mathbf{p}) K_s^+(\mathbf{p}) \Psi_m)| \\ &\leq \|(1 + |\mathbf{x}|) \chi_I\|_{L^1} \sum_{j=1}^3 \sum_{l,l'=1}^4 c_D^{l'} c_{\text{rad}}^j \left(\frac{|\partial_{p^\nu} f_s^l(\mathbf{p})|}{M} + \frac{|f_s^l(\mathbf{p})|}{M} + \frac{|f_s^l(\mathbf{p})|}{M^2} \right) (\|H_{0,m} \Psi_m\| + 1). \end{aligned} \quad (51)$$

(Second term) It is seen that

$$\begin{aligned}
& (\Phi, \partial_{p^\nu} R_{m,M}(\mathbf{p}) S_s^+(\mathbf{p}) \Psi_m) \\
&= - \sum_{l=1}^4 \partial_{p^\nu} \left(f_s^l(\mathbf{p}) \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{\Pi}(\mathbf{x}) \chi_{\Pi}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} e^{-i\mathbf{p} \cdot \mathbf{y}} (R_{m,M}(\mathbf{p}) \Phi, (\rho(\mathbf{x}) \psi_l(\mathbf{y}) \otimes \mathbb{1})) \Psi_m) d\mathbf{x} d\mathbf{y} \right) \\
&= - \sum_{l=1}^4 \left\{ (\partial_{p^\nu} f_s^l(\mathbf{p})) \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{\Pi}(\mathbf{x}) \chi_{\Pi}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} e^{-i\mathbf{p} \cdot \mathbf{y}} (R_{m,M}(\mathbf{p}) \Phi, (\rho(\mathbf{x}) \psi_l(\mathbf{y}) \otimes \mathbb{1})) \Psi_m) d\mathbf{x} d\mathbf{y} \right. \\
&\quad - i f_s^l(\mathbf{p}) \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{\Pi}(\mathbf{x}) \chi_{\Pi}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} y^\nu e^{-i\mathbf{p} \cdot \mathbf{y}} (R_{m,M}(\mathbf{p}) \Phi, (\rho(\mathbf{x}) \psi_l(\mathbf{y}) \otimes \mathbb{1})) \Psi_m) d\mathbf{x} d\mathbf{y} \\
&\quad \left. - \frac{f_s^l(\mathbf{p}) p^\nu}{\omega_M(\mathbf{p})} \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{\Pi}(\mathbf{x}) \chi_{\Pi}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} e^{-i\mathbf{p} \cdot \mathbf{y}} (R_{m,M}(\mathbf{p})^2 \Phi, (\rho(\mathbf{x}) \psi_l(\mathbf{y}) \otimes \mathbb{1})) \Psi_m) d\mathbf{x} d\mathbf{y} \right\}. \quad (52)
\end{aligned}$$

By evaluating the right-hand side of (52), we have

$$|\partial_{p^\nu} (\Phi, R_{m,M}(\mathbf{p}) S_s^+(\mathbf{p}) \Psi_m)| \leq \|(1 + |\mathbf{x}|) \chi_{\Pi}\|_{L^1} \sum_{l,l'=1}^4 (c_{\mathbf{D}}^{l'})^2 c_{\mathbf{D}}^l \left(\frac{|\partial_{p^\nu} f_s^l(\mathbf{p})|}{M} + \frac{|f_s^l(\mathbf{p})|}{M} + \frac{|f_s^l(\mathbf{p})|}{M^2} \right). \quad (53)$$

(Third term) We see that

$$\begin{aligned}
& (\Phi, \partial_{p^\nu} R_{m,M}(\mathbf{p}) T_s^+(\mathbf{p}) \Psi_m) \\
&= - \sum_{l=1}^4 \partial_{p^\nu} \left(f_s^l(\mathbf{p}) \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{\Pi}(\mathbf{x}) \chi_{\Pi}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} e^{-i\mathbf{p} \cdot \mathbf{y}} (R_{m,M}(\mathbf{p}) \Phi, (\psi_l(\mathbf{x}) \rho(\mathbf{y}) \otimes \mathbb{1})) \Psi_m) d\mathbf{x} d\mathbf{y} \right) \\
&= - \sum_{l=1}^4 \left\{ (\partial_{p^\nu} f_s^l(\mathbf{p})) \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{\Pi}(\mathbf{x}) \chi_{\Pi}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} e^{-i\mathbf{p} \cdot \mathbf{y}} (R_{m,M}(\mathbf{p}) \Phi, (\psi_l(\mathbf{x}) \rho(\mathbf{y}) \otimes \mathbb{1})) \Psi_m) d\mathbf{x} d\mathbf{y} \right. \\
&\quad - i f_s^l(\mathbf{p}) \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{\Pi}(\mathbf{x}) \chi_{\Pi}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} y^\nu e^{-i\mathbf{p} \cdot \mathbf{y}} (R_{m,M}(\mathbf{p}) \Phi, (\psi_l(\mathbf{x}) \rho(\mathbf{y}) \otimes \mathbb{1})) \Psi_m) d\mathbf{x} d\mathbf{y} \\
&\quad \left. - \frac{f_s^l(\mathbf{p}) p^\nu}{\omega_M(\mathbf{p})} \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{\Pi}(\mathbf{x}) \chi_{\Pi}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} e^{-i\mathbf{p} \cdot \mathbf{y}} (R_{m,M}(\mathbf{p})^2 \Phi, (\psi_l(\mathbf{x}) \rho(\mathbf{y}) \otimes \mathbb{1})) \Psi_m) d\mathbf{x} d\mathbf{y} \right\}. \quad (54)
\end{aligned}$$

We estimate the right-hand side of the absolute value of (54), and then,

$$|\partial_{p^\nu} (\Phi, R_{m,M}(\mathbf{p}) T_s^+(\mathbf{p}) \Psi_m)| \leq \|(1 + |\mathbf{x}|) \chi_{\Pi}\|_{L^1} \sum_{l,l'=1}^4 c_{\mathbf{D}}^l (c_{\mathbf{D}}^{l'})^2 \left(\frac{|\partial_{p^\nu} f_s^l(\mathbf{p})|}{M} + \frac{|f_s^l(\mathbf{p})|}{M} + \frac{|f_s^l(\mathbf{p})|}{M^2} \right). \quad (55)$$

From (51), (53) and (55), we have

$$\begin{aligned}
& |(\Phi, \partial_{p^\nu} (b_s(\mathbf{p}) \otimes \mathbb{1}) \Psi_m)| \\
&\leq \sum_{l=1}^4 c_+^l \left(\frac{|\partial_{p^\nu} f_s^l(\mathbf{p})|}{M} + \frac{|f_s^l(\mathbf{p})|}{M} + \frac{|f_s^l(\mathbf{p})|}{M^2} \right) \left(|\kappa_{\mathbf{I}}| \|H_{0,m} \Psi_m\| + |\kappa_{\mathbf{I}}| + 2|\kappa_{\mathbf{II}}| \right),
\end{aligned}$$

where $c_+^l = \|(1 + |\mathbf{x}|) \chi_{\Pi}\|_{L^1} \times \max \left\{ \sum_{j=1}^3 \sum_{l'=1}^4 |\alpha_{l,l'}^j| c_{\mathbf{D}}^{l'} c_{\text{rad}}^j, \sum_{l'=1}^4 (c_{\mathbf{D}}^{l'})^2 c_{\mathbf{D}}^l \right\}$. By the definition of $f_s^l(\mathbf{p}) =$

$\frac{\chi_D(\mathbf{p})u_s^l(\mathbf{p})}{\sqrt{(2\pi)^3}}$, we have

$$|(\Phi, \partial_{p^\nu}(b_s(\mathbf{p}) \otimes \mathbb{1})\Psi_m)| \leq F_{s,+}^\nu(\mathbf{p}) \left(|\kappa_I| \|H_{0,m}\Psi_m\| + |\kappa_I| + 2|\kappa_{II}| \right), \quad (56)$$

where

$$F_{s,+}^\nu(\mathbf{p}) = \frac{1}{\sqrt{(2\pi)^3}} \sum_{l=1}^4 c_+^l \left(\frac{|\partial_{p^\nu}\chi_D(\mathbf{p})|}{M} + \frac{|\chi_D(\mathbf{p})\partial_{p^\nu}u_s^l(\mathbf{p})|}{M} + \frac{|\chi_D(\mathbf{p})|}{M} + \frac{|\chi_D(\mathbf{p})|}{M^2} \right).$$

We see that (56) holds for all $\Phi \in \mathcal{F}_{\text{QED}}$ with $\|\Phi\| = 1$, and this implies that

$$\|\partial_{p^\nu}(b_s(\mathbf{p}) \otimes \mathbb{1})\Psi_m\| \leq F_{s,+}^\nu(\mathbf{p}) \left(|\kappa_I| \|H_{0,m}\Psi_m\| + |\kappa_I| + 2|\kappa_{II}| \right).$$

From Lemma 3.9, it holds that for all $0 < \varepsilon < \frac{1}{c_I|\kappa_I|}$,

$$\|H_{0,m}\Psi_m\| \leq L_\varepsilon \|H_m\Psi_m\| + R_\varepsilon \|\Psi_m\| = L_\varepsilon E_0(H_m) + R_\varepsilon.$$

Thus (i) is obtained. Similarly, (ii) is also proven in a same way as (i). ■

4.2 Photon Derivative Bound

In a similar to the Dirac field, we introduce the distribution kernel of the annihilation operator for the radiation field. For all $\Psi = \left\{ \Psi^{(n)} = \left(\Psi_1^{(n)}, \Psi_2^{(n)} \right) \right\}_{n=0}^\infty \in \mathcal{D}(H_{\text{rad},m})$, we define $a_r(\mathbf{k})$, by

$$a_r(\mathbf{k})\Psi_\rho^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n) = \delta_{r,\rho} \sqrt{n+1} \Psi_\rho^{(n+1)}(\mathbf{k}, \mathbf{k}_1, \dots, \mathbf{k}_n), \quad \rho = 1, 2.$$

It holds that

$$(\Phi, a_r(h)\Psi) = \int_{\mathbf{R}^3} h(\mathbf{k})^* (\Phi, a_r(\mathbf{k})\Psi) d\mathbf{k}, \quad \Phi \in \mathcal{F}_{\text{rad}}, \Psi \in \mathcal{D}(H_{\text{rad},m}). \quad (57)$$

Lemma 4.7 Assume (A.2). Then for all $\Phi, \Psi \in \mathcal{D}(H_{\text{rad},m})$,

- (i) $[H_{\text{rad},m}, a_r(h)]^0(\Phi, \Psi) = (\Phi, a_r(\omega_m h)\Psi),$
- (ii) $[A_j(\mathbf{x}), a_r(h)]^0(\Phi, \Psi) = -(h, h_{r,\mathbf{x}}^j)(\Phi, \Psi).$

(Proof) It holds that for all $\Phi \in \mathcal{F}_{\text{rad}}^{\text{fin}}(\mathcal{D}(\omega_m))$,

$$[H_{\text{rad},m}, a_r^\dagger(h)]\Phi = -a_r^\dagger(\omega_m h)\Phi, \quad (58)$$

$$[A_j(\mathbf{x}), a_r^\dagger(h)]\Phi = (h_{r,\mathbf{x}}^j, h)\Phi. \quad (59)$$

In a similar way to Lemma 4.4, we can prove (i) by (58) and (ii) by (59). ■

Lemma 4.8 Assume (A.1) - (A.3). Then

(i) it holds that for all $\Phi, \Psi \in \mathcal{D}(H_m)$,

$$[H_I, \mathbb{1} \otimes a_r(h)]^0(\Phi, \Psi) = \int_{\mathbf{R}^3} h(\mathbf{k})^* (\Phi, Q_r(\mathbf{k})\Psi) d\mathbf{k}.$$

Here $Q_r(\mathbf{k})$ is an operator which satisfy

$$(\Phi, Q_r(\mathbf{k})\Psi) = - \sum_{j=1}^3 \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) h_{r,\mathbf{x}}^j(\mathbf{k}) (\Phi, (\psi^\dagger(\mathbf{x}) \alpha^j \psi(\mathbf{x}) \otimes \mathbb{1}) \Psi) d\mathbf{x}$$

with $\|Q_r(\mathbf{k})\| \leq \|\chi_I\|_{L^1} \sum_{j=1}^3 \sum_{l,l'=1}^4 |h_r^j(\mathbf{k})| |\alpha_{l,l'}^j| |c_D^l| |c_D^{l'}|$.

(ii) Additionally assume (A.4) and (A.6). Then, $Q_r(\mathbf{k})\Psi$ is strongly differential for all $\mathbf{k} \in \mathbf{R}^3 \setminus O_{\text{rad}}$.

(Proof) (i) Let $\Phi \in \mathcal{D}(H_m)$ From Lemma 4.7,

$$\begin{aligned} [H_I, \mathbb{1} \otimes a_r(h)]^0(\Phi, \Psi) &= \sum_{j=1}^3 \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) [(\psi^\dagger(\mathbf{x}) \alpha^j \psi(\mathbf{x}) \otimes A_j(\mathbf{x}), \mathbb{1} \otimes a_r(h)]^0(\Phi, \Psi) d\mathbf{x} \\ &= \sum_{j=1}^3 \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) [\mathbb{1} \otimes A_j(\mathbf{x}), \mathbb{1} \otimes a_r(h)]^0(\Phi, (\psi^\dagger(\mathbf{x}) \alpha^j \psi(\mathbf{x}) \otimes \mathbb{1}) \Psi) d\mathbf{x} \\ &= - \sum_{j=1}^3 \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) (h, h_{r,\mathbf{x}}^j)(\Phi, (\psi^\dagger(\mathbf{x}) \alpha^j \psi(\mathbf{x}) \otimes \mathbb{1}) \Psi) d\mathbf{x}. \end{aligned} \quad (60)$$

We define $\ell_{r,\mathbf{k}} : \mathcal{F}_{\text{QED}} \otimes \mathcal{F}_{\text{QED}} \rightarrow \mathbf{C}$ by

$$\ell_{r,\mathbf{k}}(\Phi', \Psi') = - \sum_{j=1}^3 h_r^j(\mathbf{k}) \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} ((\psi^\dagger(\mathbf{x}) \alpha^j \psi(\mathbf{x}) \otimes \mathbb{1}) \Phi', \Psi') d\mathbf{x}$$

We see that $|\ell_{r,\mathbf{k}}(\Phi', \Psi')| \leq \|\chi_I\|_{L^1} \sum_{j=1}^3 \sum_{l,l'=1}^4 |h_r^j(\mathbf{k})| |\alpha_{l,l'}^j| |c_D^l| |c_D^{l'}| \|\Phi'\| \|\Psi'\|$. By Riesz representation theorem, we can define an operator $Q_r(\mathbf{k})$ such that $\ell_{r,\mathbf{k}}(\Phi', \Psi') = (\Phi', Q_r(\mathbf{k})\Psi')$. Then we have

$$[H_I, \mathbb{1} \otimes a_r(f)]^0(\Phi, \Psi) = \int_{\mathbf{R}^3} h(\mathbf{k})^* \ell_{r,\mathbf{k}}(\Phi, \Psi) d\mathbf{k} = \int_{\mathbf{R}^3} h(\mathbf{k})^* (\Phi, Q_r(\mathbf{k})\Psi) d\mathbf{k}.$$

Then (i) is obtained.

(ii) The strong differentiability of $Q_r(\mathbf{k})\Psi$ is proven by (A.4) and (A.6) in a similar way to Lemma 4.3, and the proof is omitted. ■

Proposition 4.9 (Photon Pull-Through Formula)

Assume (A.1) - (A.3). Then it holds that for almost everywhere $\mathbf{k} \in \mathbf{R}^3$,

$$(\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m = \kappa_I(H_m - E_0(H_m) + \omega_m(\mathbf{k}))^{-1} Q_r(\mathbf{k})\Psi_m. \quad (61)$$

(Proof) Let $\Phi \in \mathcal{D}(H_m)$. By Lemma 4.7 (i),

$$[H_m, \mathbb{1} \otimes a_r(h)]^0(\Phi, \Psi_m) = -(\Phi, (\mathbb{1} \otimes a_r(\omega_m h))\Psi_m) + \kappa_I [H_I, \mathbb{1} \otimes a_r(h)]^0(\Phi, \Psi_m).$$

It also holds that

$$[H_m, \mathbb{1} \otimes a_r(h)]^0(\Phi, \Psi_m) = ((H_m - E_0(H_m))\Phi, (\mathbb{1} \otimes a_r(h))\Psi_m).$$

Then we have

$$((H_m - E_0(H_m))\Phi, (\mathbb{1} \otimes a_r(h))\Psi_m) + (\Phi, (\mathbb{1} \otimes a_r(\omega_m h))\Psi_m) = \kappa_I [H_I, \mathbb{1} \otimes a_r(h)](\Phi, \Psi_m).$$

By Lemma 4.8,

$$\int_{\mathbf{R}^3} h(\mathbf{k})^* ((H_m - E_0(H_m) + \omega_m(\mathbf{k}))\Phi, (\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m) d\mathbf{k} = \kappa_I \int_{\mathbf{R}^3} h(\mathbf{k})^* (\Phi, Q_r(\mathbf{k})\Psi_m) d\mathbf{k}. \quad (62)$$

Note that (62) holds for all $h \in L^2(\mathbf{R}^3)$. Then we have

$$((H_m - E_0(H_m) + \omega_m(\mathbf{k}))\Phi, (\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m) = (\Phi, \kappa_I Q_r(\mathbf{k})\Psi_m), \quad (63)$$

for almost everywhere $\mathbf{k} \in \mathbf{R}^3$. In addition, (63) yields that $(\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m \in \mathcal{D}(H_m)$ and

$$(H_m - E_0(H_m) + \omega_m(\mathbf{k}))(\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m = \kappa_I Q_r(\mathbf{k})\Psi_m.$$

Thus the proof is obtained. ■

Theorem 4.10 (Photon Derivative Bounds)

Assume (A.1)-(A.4) and (A.6). Then it holds that for all $\mathbf{k} \in \mathbf{R}^3 \setminus O_{\text{rad}}$,

$$\|\partial_{k^\nu}(\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m\| \leq |\kappa_I| F_r^\nu(\mathbf{k})$$

where F_r^ν is a function which satisfy $F_r^\nu \in L^2(\mathbf{R}^3)$.

(Proof)

Let $R_m(\mathbf{k}) = (H_m - E_0(H_m) + \omega_m(\mathbf{k}))^{-1}$. From Proposition 4.9, it holds that $\mathbb{1} \otimes a_r(\mathbf{k})\Psi_m = R_m(\mathbf{k})Q_r(\mathbf{k})\Psi_m$. Then for all $\Phi \in \mathcal{F}_{\text{QED}}$,

$$\begin{aligned} &= (\Phi, \partial_{k^\nu}(\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m) \\ &= -\kappa_I \sum_{j=1}^3 \partial_{k^\nu} \left(h_r^j(\mathbf{k}) \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} (R_m(\mathbf{k})\Phi, (\psi^\dagger(\mathbf{x})\alpha^j\psi(\mathbf{x}) \otimes \mathbb{1})\Psi_m) d\mathbf{x} \right) \\ &= -\kappa_I \sum_{j=1}^3 \left\{ (\partial_{k^\nu} h_r^j(\mathbf{k})) \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} (R_m(\mathbf{k})\Phi, (\psi^\dagger(\mathbf{x})\alpha^j\psi(\mathbf{x}) \otimes \mathbb{1})\Psi_m) d\mathbf{x} \right. \\ &\quad \left. - i h_r^j(\mathbf{k}) \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) x^\nu e^{-i\mathbf{k} \cdot \mathbf{x}} (R_m(\mathbf{k})\Phi, (\psi^\dagger(\mathbf{x})\alpha^j\psi(\mathbf{x}) \otimes \mathbb{1})\Psi_m) d\mathbf{x} \right. \\ &\quad \left. - \frac{h_r^j(\mathbf{k}) k^\nu}{\omega_m(\mathbf{k})} \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} (R_m(\mathbf{k})^2 \Phi, (\psi^\dagger(\mathbf{x})\alpha^j\psi(\mathbf{x}) \otimes \mathbb{1})\Psi_m) d\mathbf{x} \right\}. \quad (64) \end{aligned}$$

By estimating the absolute value of the right-hand side of (64), we have

$$\begin{aligned} & |\partial_{k^\nu}(\Phi, R_m(\mathbf{k})Q(\mathbf{k})\Psi_m)| \\ & \leq \|(1+|\mathbf{x}|)\chi_1\|_{L^1} |\kappa_1| \sum_{j=1}^3 \sum_{l,l'=1}^4 |\alpha_{l,l'}^j| |c_D^l| |c_D^{l'}| \left(\frac{|\partial_{k^\nu} h_r^j(\mathbf{k})|}{\omega_m(\mathbf{k})} + \frac{|h_r^j(\mathbf{k})|}{\omega_m(\mathbf{k})} + \frac{|h_r^j(\mathbf{k})|}{\omega_m(\mathbf{k})^2} \right). \end{aligned}$$

From the definition of $h_r^j(\mathbf{k}) = \frac{\chi_{\text{rad}}(\mathbf{k})e_r^j(\mathbf{k})}{\sqrt{2(2\pi)^3}\omega(\mathbf{k})}$, we have

$$\partial_{k^\nu} h_r^j(\mathbf{k}) = \frac{1}{\sqrt{2(2\pi)^3}} \left(\frac{(\partial_{k^\nu} \chi_{\text{rad}}(\mathbf{k}))e_r^j(\mathbf{k})}{\omega(\mathbf{k})^{1/2}} + \frac{\chi_{\text{rad}}(\mathbf{k})\partial_{k^\nu} e_r^j(\mathbf{k})}{\omega(\mathbf{k})^{1/2}} - \frac{1}{2} \frac{\chi_{\text{rad}}(\mathbf{k})k^\nu}{\omega(\mathbf{k})^{5/2}} \right).$$

Hence, it holds that

$$|(\Phi, \partial_{k^\nu}(\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m)| \leq |\kappa_1| F_r^\nu(\mathbf{k}), \quad (65)$$

where

$$\begin{aligned} F_r^\nu(\mathbf{k}) &= \frac{\|(1+|\mathbf{x}|)\chi_1\|_{L^1}}{\sqrt{2(2\pi)^3}} \sum_{j=1}^3 \sum_{l,l'=1}^4 \left\{ |\alpha_{l,l'}^j| |c_D^l| |c_D^{l'}| \right. \\ & \quad \times \left. \left(\frac{|\partial_{k^\nu} \chi_{\text{rad}}(\mathbf{k})| + |\chi_{\text{rad}}(\mathbf{k})\partial_{k^\nu} e_r^j(\mathbf{k})| + |\chi_{\text{rad}}(\mathbf{k})|}{\omega(\mathbf{k})^{3/2}} + \frac{3}{2} \frac{|\chi_{\text{rad}}(\mathbf{k})|}{\omega(\mathbf{k})^{5/2}} \right) \right\}. \end{aligned}$$

Since (65) holds for all $\Phi \in \mathcal{F}_{\text{QED}}$, we have

$$\|\partial_{k^\nu}(\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m\| \leq |\kappa_1| F_r^\nu(\mathbf{k}).$$

The condition (A.6) yields that $F_r^\nu \in L^2(\mathbf{R}^3)$, and hence the proof is obtained. ■

5 Proof of Theorem 2.1

Let $\{\Psi_m\}_{m>0}$ be the sequence of the normalized ground state of H_m , $m > 0$. Then there exists a subsequence of $\{\Psi_{m_j}\}_{j=1}^\infty$ with $m_{j+1} < m_j$, $j \in \mathbf{N}$, such that the weak limit $\Psi_0 := \text{w-lim}_{j \rightarrow \infty} \Psi_{m_j}$ exists.

Lemma 5.1 Suppose (A.1) - (A.3). Then,

- (i) \mathcal{D}_0 is a common core of H_{QED} and H_m , $m > 0$, and H_m strongly converges to H_{QED} on \mathcal{D}_0
- (ii) $\lim_{m \rightarrow \infty} E_0(H_m) = E_0(H_{\text{QED}})$.

(i) Since \mathcal{D}_0 is a core of $H_{0,m}$, \mathcal{D}_0 is also a core of H_m . It is directly proven that $\lim_{m \rightarrow 0} H_m \Psi = H_{\text{QED}} \Psi$ for all $\Psi \in \mathcal{D}_0$.

(ii) We see that $(\Psi, H_m \Psi) \geq (\Psi, H_{\text{QED}} \Psi) \geq E_0(H_{\text{QED}})$, for all $\Psi \in \mathcal{D}_0$. Hence $\inf_{m>0} E_0(H_m) \geq E_0(H_m)$. From (i), it follows that H_m converges to H_{QED} as $m \rightarrow 0$ in the strong resolvent sense, and this yields that $\limsup_{m \rightarrow 0} E_0(H_m) \leq E_0(H_{\text{QED}})$. Hence (ii) follows. ■

From Lemma 5.1 (ii), we can set

$$E_\infty = \sup_{j \in \mathbf{N}} |E_0(H_{m_j})| < \infty.$$

Lemma 5.2 (Number Operator Bounds)

Suppose (A.1) - (A.6). Then, for all $0 < \varepsilon < \frac{1}{c_1 |\kappa_1|}$,

$$\begin{aligned} \text{(i)} \quad & \sup_{j \in \mathbf{N}} \|(N_D^{1/2} \otimes \mathbb{1}) \Psi_{m_j}\| \leq \left(\frac{L_\varepsilon}{M} E_\infty + \frac{R_\varepsilon}{M} \right)^{1/2}, \\ \text{(ii)} \quad & \sup_{j \in \mathbf{N}} \|(\mathbb{1} \otimes N_{\text{rad}}^{1/2}) \Psi_{m_j}\| \leq c_0 |\kappa_1| \left\| \frac{\chi_{\text{rad}}}{\omega^{3/2}} \right\|, \end{aligned}$$

where $c_0 = \sqrt{\frac{11}{2(2\pi)^3}} \sum_{j=1}^3 \sum_{l, l'=1}^4 |\alpha_{l, l'}^j| c_D^l c_D^{l'}$.

(Proof) (i) We see that $\|(N_D^{1/2} \otimes \mathbb{1}) \Psi_m\|^2 = (\Psi_m, (N_D \otimes \mathbb{1}) \Psi_m) \leq \|(N_D \otimes \mathbb{1}) \Psi_m\|$, and Corollary 3.10 yields that for all $0 < \varepsilon < \frac{1}{c_1 |\kappa_1|}$,

$$\|(N_D \otimes \mathbb{1}) \Psi_m\| \leq \frac{L_\varepsilon}{M} \|H_m \Psi_m\| + \frac{R_\varepsilon}{M} \|\Psi_m\| = \frac{L_\varepsilon}{M} E_0(H_m) + \frac{R_\varepsilon}{M}.$$

Hence (i) follows.

(ii) From the photon pull-through formula in Proposition 4.9, it follows that

$$\begin{aligned} (\Psi_m, (\mathbb{1} \otimes N_{\text{rad}}) \Psi_m) &= \sum_{r=1,2} \int_{\mathbf{R}^3} \|(\mathbb{1} \otimes a_r(\mathbf{k})) \Psi_m\|^2 d\mathbf{k} \\ &= |\kappa_1|^2 \sum_{r=1,2} \int_{\mathbf{R}^3} \|(H_m - E_0(H_m) + \omega_m(\mathbf{k})) Q_r(\mathbf{k}) \Psi_m\|^2 d\mathbf{k} \\ &\leq |\kappa_1|^2 \frac{11}{2(2\pi)^3} \sum_{r=1,2} \sum_{j=1}^3 \sum_{l, l'=1}^4 |\alpha_{l, l'}^j|^2 (c_D^l c_D^{l'})^2 \left(\int_{\mathbf{R}^3} \frac{|\chi_{\text{rad}}|^2}{|\mathbf{k}|^3} d\mathbf{k} \right). \end{aligned} \quad (66)$$

From (66), we obtain (ii). ■

Proposition 5.3 Assume (A.1)-(A.6). Let $F \in C_0^\infty(\mathbf{R}^3)$ which satisfy $0 \leq F \leq 1$ and $F(\mathbf{x}) = 1$ for $|\mathbf{x}| \leq 1$, and set $F_R(\mathbf{x}) = F(\frac{\mathbf{x}}{R})$. Let $\hat{\mathbf{p}} = -i\nabla_{\mathbf{p}}$ and $\hat{\mathbf{k}} = -i\nabla_{\mathbf{k}}$. Then for all $0 < \varepsilon < \frac{1}{c_1 \kappa_1}$, $R \geq 1$ and $R' \geq 1$,

$$\begin{aligned} \text{(i)} \quad & \sup_{j \in \mathbf{N}} \|(\mathbb{1} - \Gamma_{\text{f}}(F_R(\hat{\mathbf{p}})) \otimes \mathbb{1}) \Psi_{m_j}\| \leq \frac{c_{1, \varepsilon}}{\sqrt{R}}, \\ \text{(ii)} \quad & \sup_{j \in \mathbf{N}} \|(\mathbb{1} \otimes (\mathbb{1} - \Gamma_{\text{b}}(F_{R'}(\hat{\mathbf{k}}))) \Psi_{m_j}\| \leq \frac{c_2}{\sqrt{R'}}, \end{aligned}$$

where

$$c_{1,\varepsilon} = \left(4 \frac{L_\varepsilon E_\infty + R_\varepsilon}{M} \right)^{1/4} \left(\left(\frac{L_\varepsilon E_\infty + R_\varepsilon}{M} \right)^{1/2} + (L_\varepsilon E_\infty + R_\varepsilon + 1) |\kappa_I| + 2 |\kappa_{II}| \sum_{s=\pm 1/2} \sum_{\nu=1}^3 \sum_{\tau=\pm} \|F_{s,\tau}^\nu\| \right)^{1/2}$$

and

$$c_2 = |\kappa_I|^{1/2} \left(c_0 \left\| \frac{\chi_{\text{rad}}}{\omega^{3/2}} \right\| \right)^{1/2} \left(c_0 \left\| \frac{\chi_{\text{rad}}}{\omega^{3/2}} \right\| + \sum_{r=1,2} \sum_{\nu=1}^3 \|F_r^\nu\|_{L^2} \right)^{1/2}.$$

(Proof) It follows that $(\mathbb{1} - \Gamma_f(F_R(\hat{\mathbf{p}})))^2 \leq \mathbb{1} - \Gamma_f(F_R(\hat{\mathbf{p}})) \leq d\Gamma_f(1 - F_R(\hat{\mathbf{p}}))$, and then,

$$\begin{aligned} \|((\mathbb{1} - \Gamma_f(F_R(\hat{\mathbf{p}}))) \otimes \mathbb{1}) \Psi_m\|^2 &\leq (\Psi_m, (d\Gamma_f(1 - F_R(\hat{\mathbf{p}})) \otimes \mathbb{1}) \Psi_m) \\ &= \sum_{s=\pm 1/2} \left(\int_{\mathbf{R}^3} ((b_s(\mathbf{p}) \otimes \mathbb{1}) \Psi_m, (1 - F_R(\hat{\mathbf{p}})) (b_s(\mathbf{p}) \otimes \mathbb{1}) \Psi_m) d\mathbf{p} \right. \\ &\quad \left. + \int_{\mathbf{R}^3} ((d_s(\mathbf{p}) \otimes \mathbb{1}) \Psi_m, (1 - F_R(\hat{\mathbf{p}})) (d_s(\mathbf{p}) \otimes \mathbb{1}) \Psi_m) d\mathbf{p} \right). \end{aligned} \quad (67)$$

We evaluate the two terms in the right-hand side of (67). The first term is estimated as

$$\begin{aligned} &\left| \int_{\mathbf{R}^3} ((b_s(\mathbf{p}) \otimes \mathbb{1}) \Psi_m, (1 - F_R(\hat{\mathbf{p}})) (b_s(\mathbf{p}) \otimes \mathbb{1}) \Psi_m) d\mathbf{p} \right| \\ &\leq \left(\int_{\mathbf{R}^3} \|(b_s(\mathbf{p}) \otimes \mathbb{1}) \Psi_m\|^2 d\mathbf{p} \right)^{1/2} \times \left(\int_{\mathbf{R}^3} \|(1 - F_R(\hat{\mathbf{p}})) (b_s(\mathbf{p}) \otimes \mathbb{1}) \Psi_m\|^2 d\mathbf{p} \right)^{1/2} \\ &= \|(N_D^+ \otimes \mathbb{1})^{1/2} \Psi_m\| \times \left(\int_{\mathbf{R}^3} \|(1 - F_R(\hat{\mathbf{p}})) (b_s(\mathbf{p}) \otimes \mathbb{1}) \Psi_m\|^2 d\mathbf{p} \right)^{1/2}. \end{aligned}$$

It is seen that

$$\begin{aligned} &\int_{\mathbf{R}^3} \|(1 - F_R(\hat{\mathbf{p}})) (b_s(\mathbf{p}) \otimes \mathbb{1}) \Psi_m\|^2 d\mathbf{p} \\ &\leq 4 \int_{\mathbf{R}^3} \|(1 - F_R(\hat{\mathbf{p}})) \frac{1}{1 + \hat{\mathbf{p}}^2} (b_s(\mathbf{p}) \otimes \mathbb{1}) \Psi_m\|^2 d\mathbf{p} \\ &\quad + 4 \sum_{\nu=1}^3 \int_{\mathbf{R}^3} \|(1 - F_R(\hat{\mathbf{p}})) \frac{(\hat{p}^\nu)^2}{1 + \hat{\mathbf{p}}^2} (b_s(\mathbf{p}) \otimes \mathbb{1}) \Psi_m\|^2 d\mathbf{p}. \end{aligned}$$

Note that for all $\mathbf{p} \in \mathbf{R}^3$,

$$\sup_{\mathbf{p} \in \mathbf{R}^3} \left| (1 - F_R(\mathbf{p})) \frac{1}{1 + \mathbf{p}^2} \right| \leq \frac{1}{R^2}, \quad \sup_{\mathbf{p} \in \mathbf{R}^3} \left| (1 - F_R(\mathbf{p})) \frac{p^\nu}{1 + \mathbf{p}^2} \right| \leq \frac{1}{R}.$$

Then by the electron derivative bounds in Theorem 4.6 (i) and the spectral decomposition theorem, we have

$$\begin{aligned}
& \int_{\mathbf{R}^3} \|(1 - F_R(\hat{\mathbf{p}}))(b_s(\mathbf{p}) \otimes \mathbb{1})\Psi_m\|^2 d\mathbf{p} \\
& \leq \frac{4}{R^4} \int_{\mathbf{R}^3} \|(b_s(\mathbf{p}) \otimes \mathbb{1})\Psi_m\|^2 d\mathbf{p} + \frac{4}{R^2} \sum_{v=1}^3 \int_{\mathbf{R}^3} \|\partial_{p^v}(b_s(\mathbf{p}) \otimes \mathbb{1})\Psi_m\|^2 d\mathbf{p} \\
& \leq \frac{4}{R^4} \|(N_D^+ \otimes \mathbb{1})^{1/2}\Psi_m\|^2 + \frac{c_m(\varepsilon)^2}{R^2} \sum_{v=1}^3 \int_{\mathbf{R}^3} |F_{s,+}^v(\mathbf{p})|^2 d\mathbf{p},
\end{aligned}$$

where $c_m(\varepsilon) = 2(L_\varepsilon E_0(H_m) + R_\varepsilon + 1)|\kappa_I| + 4|\kappa_{II}|$. Therefore,

$$\begin{aligned}
& \left| \int_{\mathbf{R}^3} (b_s(\mathbf{p}) \otimes \mathbb{1})\Psi_m, (1 - F_R(\hat{\mathbf{p}}))(b_s(\mathbf{p}) \otimes \mathbb{1})\Psi_m d\mathbf{p} \right| \\
& \leq \|(N_D^+ \otimes \mathbb{1})^{1/2}\Psi_m\| \times \left(\frac{2}{R^2} \|(N_D^+ \otimes \mathbb{1})^{1/2}\Psi_m\| + \frac{c_m(\varepsilon)}{R} \sum_{v=1}^3 \|F_{s,+}^v\|_{L^2} \right). \quad (68)
\end{aligned}$$

In a same way as the first term, we can estimate the second term in the right-hand side of (67) by the positron derivative bounds in Theorem 4.6 (ii), and then,

$$\begin{aligned}
& \left| \int_{\mathbf{R}^3} (d_s(\mathbf{p}) \otimes \mathbb{1})\Psi_m, (1 - F_R(\hat{\mathbf{p}}))(d_s(\mathbf{p}) \otimes \mathbb{1})\Psi_m d\mathbf{p} \right| \\
& \leq \|(N_D^- \otimes \mathbb{1})^{1/2}\Psi_m\| \times \left(\frac{2}{R^2} \|(N_D^- \otimes \mathbb{1})^{1/2}\Psi_m\| + \frac{c_m(\varepsilon)}{R} \sum_{v=1}^3 \|F_{s,-}^v\|_{L^2} \right). \quad (69)
\end{aligned}$$

From (68) and (69), we have for all $R > 1$,

$$\begin{aligned}
& \|((\mathbb{1} - \Gamma_f(F_R(\hat{\mathbf{p}}))) \otimes \mathbb{1})\Psi_m\|^2 \\
& \leq \frac{1}{R} \sum_{\tau=\pm} \|(N_D^\tau \otimes \mathbb{1})^{1/2}\Psi_m\| \left(2\|(N_D^\tau \otimes \mathbb{1})^{1/2}\Psi_m\| + c_m(\varepsilon) \sum_{s=\pm 1/2} \sum_{v=1}^3 \|F_{s,\tau}^v\|_{L^2} \right). \quad (70)
\end{aligned}$$

From Lemma 5.2 (i),

$$\sup_{j \in \mathbf{N}} \|(N_D^\pm \otimes \mathbb{1})^{1/2}\Psi_{m_j}\| \leq \sup_{j \in \mathbf{N}} \|(N_D \otimes \mathbb{1})^{1/2}\Psi_{m_j}\| \leq \left(\frac{L_\varepsilon}{M} E_\infty + \frac{R_\varepsilon}{M} \right)^{1/2},$$

and we see that

$$\sup_{j \in \mathbf{N}} c_{m_j}(\varepsilon) = \sup_{j \in \mathbf{N}} (2(L_\varepsilon E_0(H_{m_j}) + R_\varepsilon + 1)|\kappa_I| + 4|\kappa_{II}|) \leq 2(L_\varepsilon E_\infty + R_\varepsilon + 1)|\kappa_I| + 4|\kappa_{II}|.$$

Hence (i) follows.

(ii) In a similar way to the proof of (i), it follows that $(\mathbb{1} - \Gamma_b(F_{R'}(\hat{\mathbf{k}})))^2 \leq \mathbb{1} - \Gamma_b(F_{R'}(\hat{\mathbf{k}})) \leq$

$d\Gamma_b(1 - F_{R'}(\hat{\mathbf{k}}))$, and hence,

$$\begin{aligned} \|(\mathbb{1} - \Gamma_b(\mathbb{1} \otimes F_{R'}(\hat{\mathbf{k}})))\Psi_m\|^2 &\leq (\Psi_m, (\mathbb{1} \otimes d\Gamma_b(1 - F_{R'}(\hat{\mathbf{k}})))\Psi_m) \\ &= \sum_{r=1,2} \int_{\mathbf{R}^3} ((\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m, (1 - F_{R'}(\hat{\mathbf{k}}))(\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m) d\mathbf{k}. \end{aligned} \quad (71)$$

We see that

$$\begin{aligned} &\left| \int_{\mathbf{R}^3} ((\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m, (1 - F_{R'}(\hat{\mathbf{k}}))(\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m) d\mathbf{k} \right| \\ &\leq \left(\int_{\mathbf{R}^3} \|(\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m\|^2 d\mathbf{p} \right)^{1/2} \times \left(\int_{\mathbf{R}^3} \|(1 - F_{R'}(\hat{\mathbf{k}}))(\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m\|^2 d\mathbf{p} \right)^{1/2} \\ &= \|(\mathbb{1} \otimes N_{\text{rad}}^{1/2})\Psi_m\| \times \left(\int_{\mathbf{R}^3} \|(1 - F_{b,R'}) (\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m\|^2 d\mathbf{k} \right)^{1/2}. \end{aligned}$$

By the photon derivative bounds in Theorem 4.10 and the spectral decomposition theorem,

$$\begin{aligned} &\int_{\mathbf{R}^3} \|(1 - F_{R'}(\hat{\mathbf{k}}))(\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m\|^2 d\mathbf{k} \\ &\leq 4 \int_{\mathbf{R}^3} \|(1 - F_{R'}(\hat{\mathbf{k}})) \frac{1}{1 + \hat{\mathbf{k}}^2} (\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m\|^2 d\mathbf{k} \\ &\quad + 4 \sum_{v=1}^3 \int_{\mathbf{R}^3} \|(1 - F_{R'}(\hat{\mathbf{k}})) \frac{(\hat{k}^v)^2}{1 + \hat{\mathbf{k}}^2} (\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m\|^2 d\mathbf{k} \\ &\leq \frac{4}{R'^4} \int_{\mathbf{R}^3} \|(\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m\|^2 d\mathbf{k} + \frac{4}{R'^2} \sum_{v=1}^3 \int_{\mathbf{R}^3} \|\partial_{k^v} (\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m\|^2 d\mathbf{k} \\ &\leq \frac{4}{R'^4} \|(\mathbb{1} \otimes N_{\text{rad}}^{1/2})\Psi_m\|^2 + \frac{4|\kappa_1|^2}{R'^2} \sum_{v=1}^3 \int_{\mathbf{R}^3} |F_r^v(\mathbf{k})|^2 d\mathbf{k}. \end{aligned}$$

Then we have

$$\begin{aligned} &\left| \int_{\mathbf{R}^3} ((\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m, (1 - F_{R'}(\hat{\mathbf{k}}))(\mathbb{1} \otimes a_r(\mathbf{k}))\Psi_m) d\mathbf{k} \right| \\ &\leq \|(\mathbb{1} \otimes N_{\text{rad}}^{1/2})\Psi_m\| \times \left(\frac{4}{R'^4} \|(\mathbb{1} \otimes N_{\text{rad}}^{1/2})\Psi_m\|^2 + \frac{4|\kappa_1|^2}{R'^2} \sum_{v=1}^3 \|F_r^v\|_{L^2}^2 \right)^{1/2}, \end{aligned}$$

and hence, for all $R' > 1$,

$$\begin{aligned} &\|((\mathbb{1} - \Gamma_b(\mathbb{1} \otimes F_{R'}(\hat{\mathbf{k}})))\Psi_m\|^2 \\ &\leq \frac{2}{R'} \|(\mathbb{1} \otimes N_{\text{rad}}^{1/2})\Psi_m\| \times \left(\|(\mathbb{1} \otimes N_{\text{rad}}^{1/2})\Psi_m\| + |\kappa_1| \sum_{r=1,2} \sum_{v=1}^3 \|F_r^v\|_{L^2} \right). \end{aligned} \quad (72)$$

From Lemma 5.2 (ii), it holds that $\sup_{j \in \mathbf{N}} \|(\mathbb{1} \otimes N_{\text{rad}}^{1/2})\Psi_{m_j}\| < c_0 |\kappa_1| \left\| \frac{\chi_{\text{rad}}}{\omega^{3/2}}, \right\|$. Therefore the proof is obtained. ■

(Proof of Theorem 2.1)

From Proposition 5.1 and a general theorem ([3] ; Lemmma 4.9), it is enough to show that $\lim_{j \rightarrow \infty} \Psi_{m_j} \neq 0$. We see that

$$\begin{aligned} \mathbb{1}_D \otimes \mathbb{1}_{\text{rad}} &= (\mathbb{1}_D - \Gamma_f(F_R(\hat{\mathbf{p}})) \otimes \mathbb{1}_{\text{rad}} + \Gamma_f(F_R(\hat{\mathbf{p}})) \otimes (\mathbb{1}_{\text{rad}} - \Gamma_b(F_{R'}(\hat{\mathbf{k}}))) \\ &\quad + \Gamma_f(F_R(\hat{\mathbf{p}})) \otimes (F_{R'}(\hat{\mathbf{k}}))E_{N_{\text{rad}}}([0, n]) + \Gamma_f(F_R(\hat{\mathbf{p}})) \otimes (\Gamma_b(F_{R'}(\hat{\mathbf{k}}))E_{N_{\text{rad}}}([n+1, \infty))). \end{aligned}$$

Then by Proposition 5.3, we have for all $0 < \varepsilon < \frac{1}{c_1|\kappa_1|}$, $R > 1$ and $R' > 1$,

$$\begin{aligned} &\|(\Gamma_f(F_R(\hat{\mathbf{p}})) \otimes (\Gamma_b(F_{R'}(\hat{\mathbf{k}}))E_{N_{\text{rad}}}([0, n]))) \Psi_{m_j}\| \\ &\geq 1 - \left(\|((\mathbb{1}_D - \Gamma_f(F_R(\hat{\mathbf{p}}))) \otimes \mathbb{1}_{\text{rad}}) \Psi_{m_j}\| \right. \\ &\quad \left. + \|(\mathbb{1}_D \otimes (\mathbb{1}_{\text{rad}} - \Gamma_b(F_{R'}(\hat{\mathbf{k}}))) \Psi_{m_j}\| + \|(\mathbb{1}_D \otimes E_{N_{\text{rad}}}([n+1, \infty))) \Psi_{m_j}\| \right). \\ &\geq 1 - \left(\frac{c_{1,\varepsilon}}{\sqrt{R}} + \frac{c_2}{\sqrt{R'}} + \|(\mathbb{1}_D \otimes E_{N_{\text{rad}}}([n+1, \infty))) \Psi_{m_j}\| \right), \end{aligned}$$

It is seen that

$$\sqrt{n+1} \|(\mathbb{1}_D \otimes E_{N_{\text{rad}}}([n+1, \infty))) \Psi_{m_j}\| \leq \|(\mathbb{1}_D \otimes N_{\text{rad}}^{1/2}) \Psi_{m_j}\| \leq c_0|\kappa_1| \left\| \frac{\chi_{\text{rad}}}{\omega^{3/2}} \right\|.$$

Then from Lemma 5.2 (ii), we have

$$\sup_{j \geq 1} \|(\mathbb{1}_D \otimes E_{N_{\text{rad}}}([n+1, \infty))) \Psi_{m_j}\| \leq \frac{c_3}{(n+1)^{1/2}}.$$

where $c_3 = c_0|\kappa_1| \left\| \frac{\chi_{\text{rad}}}{\omega^{3/2}} \right\|$. Then it follows that

$$\|\Gamma_f(F_R(\hat{\mathbf{p}})) \otimes (\Gamma_b(F_{R'}(\hat{\mathbf{k}}))E_{N_{\text{rad}}}([0, n])) \Psi_{m_j}\| \geq 1 - \left(\frac{c_{1,\varepsilon}}{R} + \frac{c_2}{R'} + \frac{c_3}{(n+1)^{1/2}} \right). \quad (73)$$

We also see that

$$\begin{aligned} &\|\Gamma_f(F_R(\hat{\mathbf{p}})) \otimes (F_{R'}(\hat{\mathbf{k}}))E_{N_{\text{rad}}}([0, n])) \Psi_{m_j}\|^2 \\ &= ((H_0 + 1)E_{N_{\text{rad}}}([0, n]) \Psi_{m_j}, (H_0 + 1)^{-1} (\Gamma_f(F_R(\hat{\mathbf{p}}))^2 \otimes (\Gamma_b(F_{R'}(\hat{\mathbf{k}}))^2)E_{N_{\text{rad}}}([0, n]))) \Psi_{m_j}) \\ &\leq \|(H_0 + 1) \Psi_{m_j}\| \|(H_0 + 1)^{-1} (\Gamma_f(F_R(\hat{\mathbf{p}}))^2 \otimes (\Gamma_b(F_{R'}(\hat{\mathbf{k}}))^2)E_{N_{\text{rad}}}([0, n])) \Psi_{m_j}\|. \end{aligned}$$

We see that $\|(H_0 + 1) \Psi_{m_j}\| \leq \|H_0 \Psi_{m_j}\| + 1 \leq \|H_{0,m_j} \Psi_{m_j}\| + 1$ and Lemma 3.9 yields that

$$\|H_{0,m_j} \Psi_{m_j}\| \leq \frac{L_\varepsilon}{M} \|H_{m_j} \Psi_{m_j}\| + \frac{R_\varepsilon}{M} \leq \frac{L_\varepsilon}{M} E_\infty + \frac{R_\varepsilon}{M}.$$

Then we have

$$\begin{aligned} &\|\Gamma_f(F_R(\hat{\mathbf{p}})) \otimes (F_{R'}(\hat{\mathbf{k}}))E_{N_{\text{rad}}}([0, n])) \Psi_{m_j}\| \\ &\leq c_{4,\varepsilon} \|(H_0 + 1)^{-1} ((\Gamma_f(F_R(\hat{\mathbf{p}}))^2 \otimes (\Gamma_b(F_{R'}(\hat{\mathbf{k}}))^2)E_{N_{\text{rad}}}([0, n]))) \Psi_{m_j}\|^{1/2}, \quad (74) \end{aligned}$$

where $c_{4,\varepsilon} = (\frac{L_\varepsilon E_\infty + R_\varepsilon}{M} + 1)^{1/2}$. From (73) and (74)

$$\|(H_0 + 1)^{-1} \Gamma_f(F_R(\hat{\mathbf{p}})^2) \otimes (\Gamma_b(F_{R'}(\hat{\mathbf{k}})^2 E_{N_{\text{rad}}}([0, n])) \Psi_{m_j}\| \geq \frac{1}{c_{4,\varepsilon}} \left(1 - \left(\frac{c_{1,\varepsilon}}{\sqrt{R}} + \frac{c_2}{\sqrt{R'}} + \frac{c_3}{(n+1)^{1/2}} \right) \right)^2$$

Since $(H_0 + 1)^{-1} (\Gamma_f(F_R(\hat{\mathbf{p}})^2) \otimes (\Gamma_b(F_{R'}(\hat{\mathbf{k}})^2 E_{N_{\text{rad}}}([0, n])))$ is compact, we have

$$\|(H_0 + 1)^{-1} \Gamma_f(F_R(\hat{\mathbf{p}})^2) \otimes (\Gamma_b(F_{R'}(\hat{\mathbf{k}})^2 E_{N_{\text{rad}}}([0, n])) \Psi_0\| \geq \frac{1}{c_{4,\varepsilon}^2} \left(1 - \left(\frac{c_{1,\varepsilon}}{\sqrt{R}} + \frac{c_2}{\sqrt{R'}} + \frac{c_3}{(n+1)^{1/2}} \right) \right)^2, \quad (75)$$

where we set $\Psi_0 = \text{w-lim}_{j \rightarrow \infty} \Psi_{m_j}$. Then for sufficiently large $R > 0$, $R' > 0$ and $n > 0$, the right-hand side of (75) is greater than zero, and hence $\Psi_0 \neq 0$.

(Multiplicity) Assume $\dim \ker (H - E_0(H_{\text{QED}})) = \infty$. Let Ψ_l , $l \in \mathbf{N}$, be the ground states. Let \mathcal{M} be the closure of the linear hull of $\{\Psi_l\}_{l=0}^\infty$. Then \mathcal{H}_{QED} is decomposed as $\mathcal{H}_{\text{QED}} = \mathcal{M} \oplus \mathcal{M}^\perp$. Let $\{\Phi_l\}_{l=0}^\infty$ be a complete orthogonal system of \mathcal{M}^\perp . We can set a complete orthonormal system $\{\Xi_l\}_{l=0}^\infty$ of \mathcal{H}_{QED} by $\Xi_{2l-1} = \Psi_l$ and $\Xi_{2l} = \Phi_l$ for all $l \in \mathbf{N}$. Since $\{\Xi_l\}_{l=0}^\infty$ is a complete orthonormal system, $\text{w-lim}_{l \rightarrow \infty} \Xi_l = 0$. On the other hand, Ξ_{2l-1} is ground state for all $l \in \mathbf{N}$, and hence $H_{\text{QED}} \Xi_{2l-1} = E_0(H_{\text{QED}}) \Xi_{2l-1}$. In a same argument of the proof of the existence of the ground state, we have

$$\|(H_0 + 1)^{-1} \Gamma_f(F_R(\hat{\mathbf{p}})^2) \otimes (\Gamma_b(F_{R'}(\hat{\mathbf{k}})^2 E_{N_{\text{rad}}}([0, n])) \Xi_{2l-1}\| \geq \frac{1}{\tilde{c}_{4,\varepsilon}^2} \left(1 - \left(\frac{\tilde{c}_{1,\varepsilon}}{\sqrt{R}} + \frac{c_2}{\sqrt{R'}} + \frac{c_3}{(n+1)^{1/2}} \right) \right)^2, \quad (76)$$

where $\tilde{c}_{1,\varepsilon}$ and $\tilde{c}_{4,\varepsilon}$ are the constants $c_{1,\varepsilon}$ and $c_{4,\varepsilon}$ replacing E_∞ with $E_0(H_{\text{QED}})$. Then by taking sufficiently large $R > 0$, $R' > 0$ and $n > 0$, we have $\text{w-lim}_{l \rightarrow \infty} \Xi_{2l-1} \neq 0$, but this is contradict to $\text{w-lim}_{l \rightarrow \infty} \Xi_l = 0$. Hence $\dim \ker (H_{\text{QED}} - E_0(H_{\text{QED}})) < \infty$. ■

[Concluding remarks]

(1) The case of Massless Dirac field

It is not realistic model, but we can consider the system of a massless Dirac field coupled to the radiation field. In such a case, by replacing (A.5) with similar conditions to (A.6), we can also prove the existence of the ground state in a same ways as H_{QED} .

(2) Infrared divergent problem

For some systems of particles coupled to massless Bose fields, the existence of the ground states without infrared regularity conditions was obtained (refer to e.g., Bach-Fröhlich-Sigal [6], Griesemer-Lieb-Loss [16] and Hasler-Herbst [18]), and non-existence of the ground states for other other systems was also investigated (see e.g., Arai-Hirokawa-Hiroshima [4]). To prove the existence or non-existence of the ground state of H_{QED} without infrared regularity conditions is left for future study.

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